GVI in Function Spaces

Gaussian Measures meet Bayesian Deep Learning

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25$^{\text{th}}$ of May 2022

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   - Bayesian Deep Learning
   - Variational Inference in Function Spaces
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Bayesian Deep Learning

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Let $\mathcal{D} := \{(\mathrm{x_n}, \mathrm{y_n}) \,|\, \mathrm{n} = 1, \ldots, \mathrm{N}\} \subset \mathcal{X} \times \mathcal{Y}$ be data.

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$$\mathrm{Y} = \mathrm{f}(\mathrm{x}) + \mathcal{N}(0, \sigma^2), \tag{1}$$

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- Predictions for arbitrary $\mathrm{x}^* \in \mathcal{X}$ follow from Bayes rule:

$$\begin{aligned} \mathrm{p}(\mathrm{y}^*|\mathcal{D}) &= \int \mathrm{p}\big(\mathrm{y}^*|\mathrm{w}\big)\mathrm{p}(\mathrm{w}|\mathcal{D})\,\mathrm{dw} &(2)\\ &= \int \mathrm{p}\big(\mathrm{y}^*|\mathrm{f}(\mathrm{x}^*; \mathrm{w})\big)\mathrm{p}(\mathrm{w}|\mathcal{D})\,\mathrm{dw} &(3) \end{aligned}$$

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- Langevin Dynamics [Welling and Teh, 2011]

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- What priors on the function space are induced by $\mathrm{p}(\mathrm{w})$?

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  $\rightarrow$ use generalised variational inference in infinite dimensions!

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for inference where $\mathbb{D}$ is an arbitrary divergence.

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  for inference where $\mathbb{D}$ is an arbitrary divergence.
- How to define priors and variational measures $\mathbb{P}^{\mathrm{F}}$ and $\mathbb{Q}^{\mathrm{F}}$ in infinite dimensions?

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Definition (Gaussian Random Element)

A random mapping $\mathrm{F} : \Omega \to \mathrm{H}$ is called Gaussian random element (GRE) if and only if

$$\langle \mathrm{F}, \mathrm{h} \rangle : \Omega \to \mathbb{R} \tag{9}$$

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and the (linear) covariance operator $\mathrm{C} : \mathrm{H} \to \mathrm{H}$ of F is defined as

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- $\mathrm{C} : \mathrm{H} \to \mathrm{H}$ of a GRE is a positive self-adjoint trace-class operator.
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Definition (Gaussian Measure)

Let $\mathrm{F} \sim \mathcal{N}(\mathrm{m}, \mathrm{C})$ be a GRE. Then P defined as

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for any (measurable) $\mathrm{A} \subset \mathrm{H}$ is called a Gaussian measure.

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Gaussian Wasserstein Inference:

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- $\mathrm{E} = \mathrm{L}^2(\mathcal{X}, \rho, \mathbb{R}) := \big\{\mathrm{f} : \mathcal{X} \to \mathbb{R} \mid \int |\mathrm{f}(\mathrm{x})|^2 \, \mathrm{d}\rho(\mathrm{x}) < \infty\big\}$ with ρ input distribution on $\mathcal{X}$

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with:

$$\mathrm{C}_{\mathrm{P}}\mathrm{g} := \int \mathrm{k}(\cdot, \mathrm{x}')\mathrm{g}(\mathrm{x}')\,\mathrm{d}\rho(\mathrm{x}'), \qquad \mathrm{C}_{\mathrm{Q}}\mathrm{g} := \int \mathrm{r}(\cdot, \mathrm{x}')\mathrm{g}(\mathrm{x}')\,\mathrm{d}\rho(\mathrm{x}') \tag{15}$$

for all $\mathrm{g} \in \mathrm{L}^2(\mathcal{X}, \rho, \mathbb{R})$ where k and r are trace-class kernels.

Regression

Regression

For regression:

$$p(\mathrm{y}|F) := \prod_{n=1}^{N} p(y_n|F) := \prod_{n=1}^{N} \mathcal{N}(y_n \,|\, F(x_n), \sigma^2), \tag{16}$$

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where $\sigma^2 > 0$.
The Wasserstein distance is tractable [Gelbrich, 1990]:

$$W_2^2(P, Q) = \|m_P - m_Q\|_2^2 + \mathrm{tr}(C_P) + \mathrm{tr}(C_Q) - 2 \cdot \mathrm{tr}\left[\left(C_P^{1/2} C_Q C_P^{1/2}\right)^{1/2}\right], \tag{17}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of an operator and $C_P^{1/2}$ is the square root of the positive, self-adjoint operator C_P.

Approximation of Wasserstein distance

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Further:

$$\mathrm{tr}(C_P) = \int k(x,x)\, d\rho(x) \approx \frac{1}{N}\sum_{n=1}^{N} k(x_n, x_n) \tag{20}$$

$$\mathrm{tr}(C_Q) = \int r(x,x)\, d\rho(x) \approx \frac{1}{N}\sum_{n=1}^{N} r(x_n, x_n) \tag{21}$$

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The last term can be approximated as

$$\mathrm{tr}\left[\left(\mathrm{C}_{\mathrm{P}}^{1/2}\mathrm{C}_{\mathrm{Q}}\mathrm{C}_{\mathrm{P}}^{1/2}\right)^{1/2}\right] \approx \frac{1}{\sqrt{\mathrm{N}\mathrm{N}_{\mathrm{S}}}}\sum_{\mathrm{s}=1}^{\mathrm{N}_{\mathrm{S}}}\sqrt{\lambda_{\mathrm{s}}\left(\mathrm{r}(\mathrm{X}_{\mathrm{S}},\mathrm{X})\mathrm{k}(\mathrm{X},\mathrm{X}_{\mathrm{S}})\right)}, \tag{22}$$

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$$\mathrm{tr}\Big[\big(C_P^{1/2} C_Q C_P^{1/2}\big)^{1/2}\Big] \approx \frac{1}{\sqrt{N N_S}} \sum_{s=1}^{N_S} \sqrt{\lambda_s\big(r(X_S, X) k(X, X_S)\big)}, \tag{22}$$

where $X_S := (x_{S,1}, \ldots, x_{S,N_S})$, $N_S \in \mathbb{N}$ with:

$$X_{S,1}, \ldots, X_{S,N_S} \overset{\text{ind.}}{\sim} \hat{\rho} \tag{23}$$

$$r(X_S, X) := \big(r(x_{S,s}, x_n)\big)_{s,n} \tag{24}$$

$$k(X, X_S) := \big(k(x_n, x_{S,s})\big)_{n,s} \tag{25}$$

and $\lambda_s\big(r(X_S, X)k(X, X_S)\big)$ denotes the s-th eigenvalue of the matrix $r(X_S, X)k(X, X_S) \in \mathbb{R}^{N_S \times N_S}$.

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  $\longrightarrow$ very scalable for typical $\mathrm{N_S}, \mathrm{N_B} << \mathrm{N}$, e.g. $\mathrm{N_S} = \mathrm{N_B} = 100$

Recovering Other Methods

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- Stochastic Variational Gaussian processes (SVGP) [Titsias, 2009]:

$$m_Q(x) := m_P(x) + \sum_{m=1}^{M} \beta_m k_m(x) \tag{30}$$

$$r(x, x') := k(x, x') - k_Z(x)^T k(Z, Z)^{-1} k_Z(x) + k_Z(x)^T \Sigma k_Z(x), \tag{31}$$

where $\beta = (\beta_1, \dots, \beta_M) \in \mathbb{R}^M$ and $\Sigma \in \mathbb{R}^{M \times M}$ are variational parameters. Further $Z = (Z_1, \dots, Z_M)$ with $\{Z_m\}_{m=1}^M \overset{\text{iid}}{\sim} \widehat{\rho}$.

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- Decoupled SVGPs [Cheng and Boots, 2017]: Same kernel r as in SVGP but mean

$$\mathrm{m_Q(x) := m_P(x) + \sum_{n=1}^{\tilde{N}} \beta_n k_n(x),} \tag{32}$$

  where $\widetilde{N} > M$.

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GWI-net

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Use neural net for posterior mean

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- Let $L \in \mathbb{N}$ be the number of hidden layers.
- Let D_ℓ, $\ell = 0, \ldots, L+1$ be the width of layer ℓ with $D_0 := D$.
- Define $g^1(x) := W^1 x + b^1$ and further

$$h^\ell(x) := \phi\big(g^\ell(x)\big), \tag{33}$$

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and the SVGP kernel r in (31) for the posterior covariance.

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Contents

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Toy Examples: GWI-net

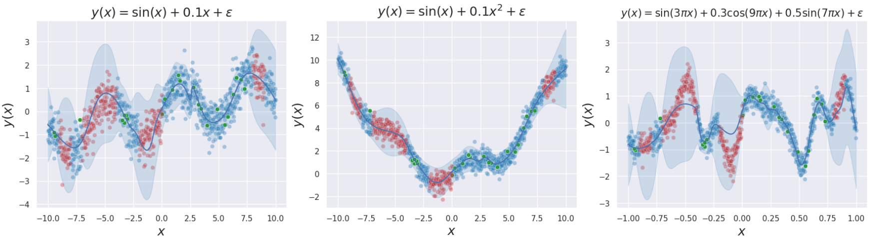

Figure 1: ■ : Training data ■ : Unseen data ■ : Inducing points
We use $N = 1000$ equidistant points and add white noise with $\epsilon \sim \mathcal{N}(0, 0.5^2)$. The plot shows $m_Q(x) \pm 1.96\sqrt{\mathbb{V}[Y^*(x)|Y]}$ where $\mathbb{V}[Y^*(x)|Y]$ is the posterior predictive variance given as $r(x, x) + \sigma^2$.

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UCI Regression

UCI Regression

Dataset	N	D	GWI SVGP	GWI DNN-SVGP	FVI	VIP-BNN	VIP-NP	BBB	VDO	$\alpha = 0.5$	FBNN	**EXACT GP**
BOSTON	506	13	2.8±0.31	**2.27±0.06**	2.33±0.04	2.45±0.04	2.45±0.03	2.76±0.04	2.63±0.10	2.45±0.02	2.30±0.10	2.46±0.04
CONCRETE	1030	8	3.24±0.09	**2.64±0.06**	2.88±0.06	3.02±0.02	3.13±0.02	3.28±0.01	3.23±0.01	3.06±0.03	3.09±0.01	3.05±0.02
ENERGY	768	8	1.81±0.19	0.91±0.12	0.58±0.05	**0.56±0.04**	0.60±0.03	2.17±0.02	1.13±0.02	0.95±0.09	0.68±0.02	0.54±0.02
KIN8NM	8192	8	-0.86±0.38	**-1.2±0.03**	-1.15±0.01	-1.12±0.01	-1.05±0.00	-0.81±0.01	-0.83±0.01	-0.92±0.02	N/A±0.00	N/A±0.00
POWER	9568	4	3.35±0.22	2.74±0.02	**2.69±0.00**	2.92±0.00	2.90±0.00	2.83±0.01	2.88±0.00	2.81±0.00	N/A±0.00	N/A±0.00
PROTEIN	45730	9	**2.84±0.04**	2.87±0.0	2.85±0.00	2.87±0.00	2.96±0.02	3.00±0.00	2.99±0.00	2.90±0.00	N/A±0.00	N/A±0.00
RED WINE	1588	11	0.97±0.02	**0.76±0.08**	0.97±0.06	0.97±0.02	1.20±0.04	1.01±0.02	0.97±0.02	1.01±0.02	1.04±0.01	0.26±0.03
YACHT	308	6	2.37±0.55	0.29±0.1	0.59±0.11	**-0.02±0.07**	0.59±0.13	1.11±0.04	1.22±0.18	0.79±0.11	1.03±0.03	0.10±0.05
NAVAL	11934	16	**-7.25±0.08**	-6.76±0.1	-7.21±0.06	-5.62±0.04	-4.11±0.00	-2.80±0.00	-2.80±0.00	-2.97±0.14	-7.13±0.02	N/A±0.00
Mean Rank			5.5	**2.06**	2.22	3.33	4.94	7	6.11	4.83		

Table 1: The table shows the average test NLL on several UCI regression datasets. We train on random 90% of the data and predict on 10%. This is repeated 10 times and we report mean and standard deviation. The results for our competitors are taken from Ma and Hernández-Lobato [2021].

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Classification

Classification

Model	FMNIST			CIFAR 10		
	Accuracy	NLL	OOD-AUC	Accuracy	NLL	OOD-AUC
GWI-net	**93.25 ±0.09**	**0.250 ±0.00**	**0.959 ±0.01**	**83.82 ±0.00**	**0.553 ±0.00**	0.618 ±0.00
FVI	91.60±0.14	0.254±0.05	0.956±0.06	77.69 ±0.64	0.675±0.03	0.883±0.04
MFVI	91.20±0.10	0.343±0.01	0.782±0.02	76.40±0.52	1.372±0.02	0.589±0.01
MAP	91.39±0.11	0.258±0.00	0.864±0.00	77.41±0.06	0.690±0.00	0.809±0.01
KFAC-LAPLACE	84.42±0.12	0.942±0.01	0.945±0.00	72.49±0.20	1.274±0.01	0.548±0.01
RITTER et al.	91.20±0.07	0.265±0.00	0.947±0.00	77.38±0.06	0.661±0.00	0.796±0.00

Table 2: We report average accuracy, NLL and OOD-AUC on test data for 10 different train/test splits.

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