Wavelet localisation of isotropic random fields on spherical manifolds and cosmological implications

Scale-discretised wavelets on the sphere and ball

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Outline

1. Scale-discretised wavelets on the sphere and ball
2. Sampling theory and fast algorithms
3. E/B separation for CMB polarization and cosmic shear

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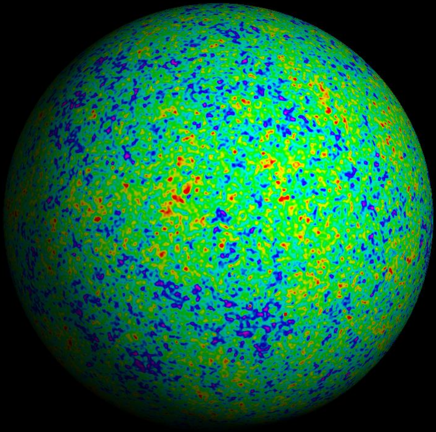
2. Sampling theory and fast algorithms
3. E/B separation for CMB polarization and cosmic shear

Cosmic microwave background (CMB) observed on 2D sphere



Credit: WMAP

Wavelets on the sphere
Dilation and translation

- Construct wavelet atoms from affine transformations (dilation, translation) on the sphere of a mother wavelet.

- The natural extension of translations to the sphere are rotations. Rotation of a function $f \in \mathrm{L}^2(\mathbb{S}^2)$ on the sphere is defined by

$$[\mathcal{R}(\rho)f](\omega) = f(\mathrm{R}_\rho^{-1}\omega), \quad \omega = (\theta, \varphi) \in \mathbb{S}^2, \quad \rho = (\alpha, \beta, \gamma) \in \mathrm{SO}(3) .$$

- How define dilation on the sphere?
  - Stereographic projection
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  - Harmonic dilation wavelets
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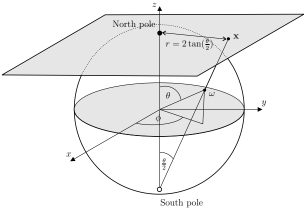

Figure: Stereographic projection

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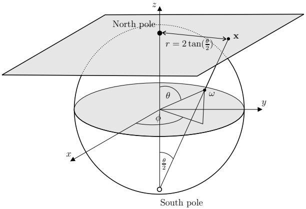

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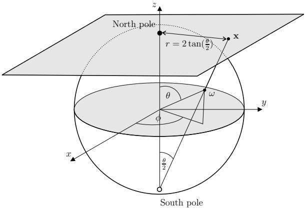

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Scale-discretised wavelets on the sphere

Directional spin wavelets

- *Exact reconstruction with directional wavelets on the sphere*
  Wiaux, McEwen, Vandergheynst, Blanc (2008)

- Extend to functions of arbitrary spin.
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  Spin s signals transform under local rotations of χ by $\boxed{{}_s f' = \mathrm{e}^{-\mathrm{i}s\chi}\ {}_s f}$.

- Why directional wavelets?
  - Peaks of isotropic random fields elongated
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Scale-discretised wavelets on the sphere

Axisymmetric kernel construction

- Spin scale-discretised wavelet ${}_s\Psi^j$ constructed in separable form in harmonic space:

$$ {}_s\Psi^j_{\ell m} = \underbrace{\kappa^j(\ell)}_{\text{axisymmetric}} \times \underbrace{\zeta_{\ell m}}_{\text{directional}} . $$

- Admissible wavelets constructed to satisfy a partition of the identity:

$$ \underbrace{|{}_s\Phi_{\ell 0}|^2}_{\text{scaling function}} + \sum_{j=0}^{J} \sum_{m=-\ell}^{\ell} \underbrace{|{}_s\Psi^j_{\ell m}|^2}_{\text{wavelet}} = 1 , \quad \forall \ell . $$

- Axisymmetric wavelet kernels $\kappa^j(\ell)$: smooth, infinitely differentiable (Schwarz) functions with compact support.

  (Similar but different to needlets.)

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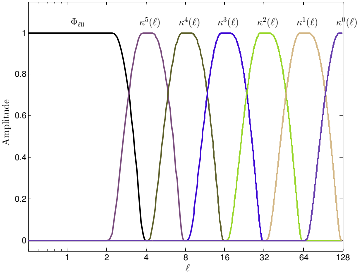

Figure: Harmonic tiling $\kappa^j(\ell)$.

- Axisymmetric wavelet kernels $\kappa^j(\ell)$: smooth, infinitely differentiable (Schwarz) functions with compact support.

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Directional kernel construction

- Consider directional auto-correlation:

$$\Gamma^{(j)}(\Delta\gamma = \gamma' - \gamma) \equiv \langle \Psi^{(j)}_{\gamma},\ \Psi^{(j)}_{\gamma'} \rangle ,$$

- Impose auto-correlation of the form:

$$\Gamma^{(j)}(\Delta\gamma) = \sum_{\ell=0}^{\infty} \left|\kappa^{(j)}(\ell)\right|^2 \cos^p(\Delta\gamma) .$$

- Recover directional wavelet kernel:

$$\boxed{\zeta_{\ell m} = \eta\, \upsilon \sqrt{\frac{1}{2^p}\binom{p}{(p-m)/2}}} ,$$

where

$$\eta = \begin{cases} 1, & \text{if } N-1 \text{ even} \\ \mathrm{i}, & \text{if } N-1 \text{ odd} \end{cases} , \quad \upsilon = [1-(-1)^{N+m}]/2 = \begin{cases} 0, & \text{if } N+m \text{ even} \\ 1, & \text{if } N+m \text{ odd} \end{cases}$$

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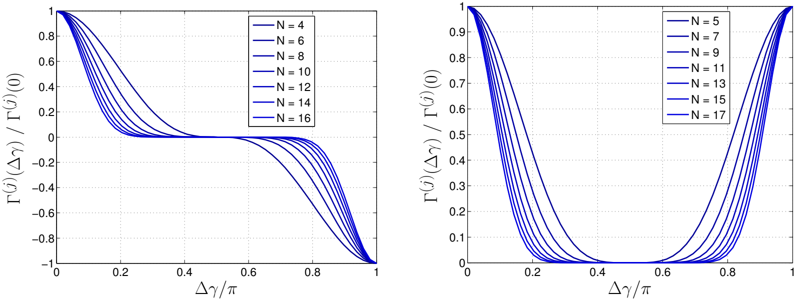
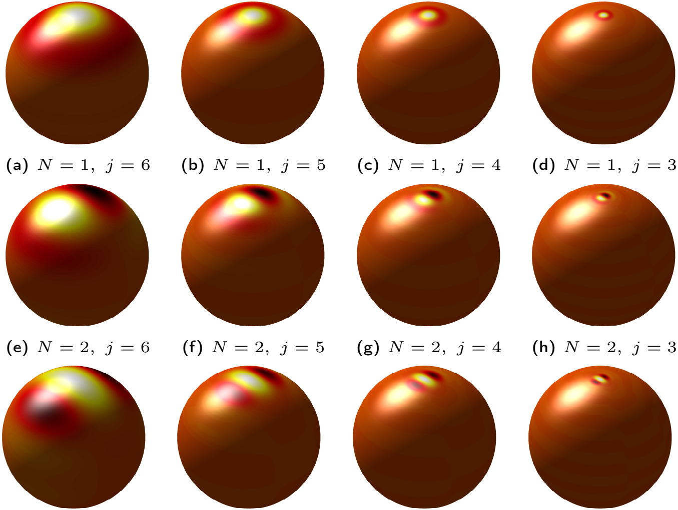


(a) Odd $N-1$ (b) Even $N-1$

Figure: Directional auto-correlation for even and odd $N-1$.

Scale-discretised wavelets on the sphere

Scalar wavelets



(a) $N = 1,\ j = 6$ (b) $N = 1,\ j = 5$ (c) $N = 1,\ j = 4$ (d) $N = 1,\ j = 3$

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Figure: Scalar scale-discretised wavelets on the sphere.

Scale-discretised wavelets on the sphere

Spin wavelets

(a) Real$({}_s\Psi^j)$ (b) Imag$({}_s\Psi^j)$ (c) Abs$({}_s\Psi^j)$

Figure: Spin scale-discretised wavelets on the sphere.

Scale-discretised wavelets on the sphere

Forward and inverse transform (i.e. analysis and synthesis)

- The spin scale-discretised wavelet transform is given by the usual projection onto each wavelet:

$$W^{{}_s\Psi^j}(\rho) = \langle {}_s f, \mathcal{R}_\rho \, {}_s\Psi^j \rangle = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \, {}_s f(\omega) (\mathcal{R}_\rho \, {}_s\Psi^j)^*(\omega) \, .$$

projection

- Framework applied for functions of any spin.
- Wavelet coefficients are scalar and not spin.
- Wavelet coefficients live in $\mathrm{SO}(3) \times \mathbb{Z}$; thus, directional structure is naturally incorporated.

- The original function may be recovered exactly in practice from the wavelet (and scaling) coefficients:

$${}_s f(\omega) = \sum_{j=0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) W^{{}_s\Psi^j}(\rho) (\mathcal{R}_\rho \, {}_s\Psi^j)(\omega) \, .$$

finite sum

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finite sum — wavelet contribution

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Steerability

- By imposing an azimuthal band-limit N, we recover steerable wavelets.
- By the linearity of the wavelet transform, steerability extends to wavelet coefficients:

$$W^{{}_s\Psi^j}(\alpha, \beta, \gamma) = \sum_{g=0}^{M-1} z(\gamma - \gamma_g)\, W^{{}_s\Psi^j}(\alpha, \beta, \gamma_g)\,.$$

steerability

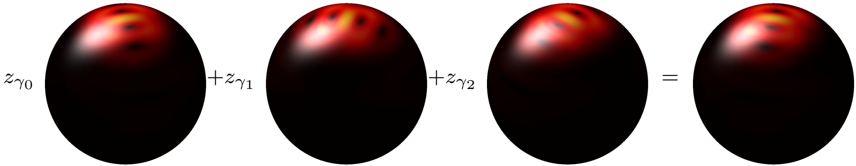

Figure: Steered wavelet computed from basis wavelets.

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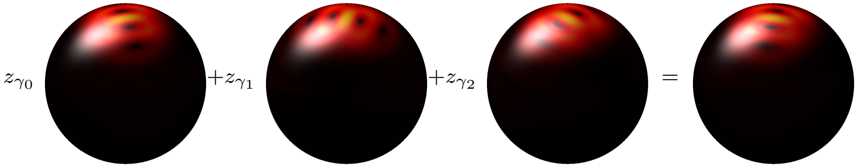

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Localisation of Gaussian random fields

Wavelet localisation (McEwen, Durastanti, Wiaux 2017)

Directional scale-discretised wavelets $\Psi \in \mathrm{L}^2(\mathbb{S}^2)$, defined on the sphere $\mathbb{S}^2$ and centred on the North pole, satisfy the localisation bound:

$$\left|\Psi^{(j)}(\theta, \varphi)\right| \leq \frac{C_1^{(j)}}{\left(1 + C_2^{(j)}\,\theta\right)^{\xi}}$$

(there exist strictly positive constants $C_1^{(j)}, C_2^{(j)} \in \mathbb{R}_*^+$ for any $\xi \in \mathbb{R}_*^+$). Follows from theorem by Geller & Mayeli (2009).

Wavelet asymptotic uncorrelation (McEwen, Durastanti, Wiaux 2017)

For Gaussian random fields on the sphere, directional scale-discretised wavelet coefficients are asymptotically uncorrelated. The directional wavelet correlation satisfies the bound:

$$\Xi^{(jj')}(\rho_1, \rho_2) \leq \frac{C_1^{(j)}}{\left(1 + C_2^{(j)} \beta\right)^{\xi}},$$

where $\beta \in [0, \pi)$ is an angular separation between Euler angles ρ_1 and ρ_2 (there exist strictly positive constants $C_1^{(j)}, C_2^{(j)} \in \mathbb{R}_*^+$ for any $\xi \in \mathbb{R}_*^+$, $\xi \geq 2N$, where N is the azimuthal band-limit of the wavelet and $|j - j'| < 2$). Follows from theorem by Geller & Mayeli (2009).

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(there exist strictly positive constants $C_1^{(j)}, C_2^{(j)} \in \mathbb{R}_*^+$ for any $\xi \in \mathbb{R}_*^+$). Follows from theorem by Geller & Mayeli (2009).

Wavelet asymptotic uncorrelation (McEwen, Durastanti, Wiaux 2017)

For Gaussian random fields on the sphere, directional scale-discretised wavelet coefficients are asymptotically uncorrelated. The directional wavelet correlation satisfies the bound:

$$\Xi^{(jj')}(\rho_1,\rho_2) \leq \frac{C_1^{(j)}}{\left(1 + C_2^{(j)}\beta\right)^{\xi}} \, ,$$

where $\beta \in [0,\pi)$ is an angular separation between Euler angles ρ_1 and ρ_2 (there exist strictly positive constants $C_1^{(j)}, C_2^{(j)} \in \mathbb{R}_*^+$ for any $\xi \in \mathbb{R}_*^+$, $\xi \geq 2N$, where N is the azimuthal band-limit of the wavelet and $\left|j - j'\right| < 2$). Follows from theorem by Geller & Mayeli (2009).

Scale-discretised wavelets on the sphere

Parseval frame

Parseval frame property (McEwen, Durastanti, Wiaux 2017)

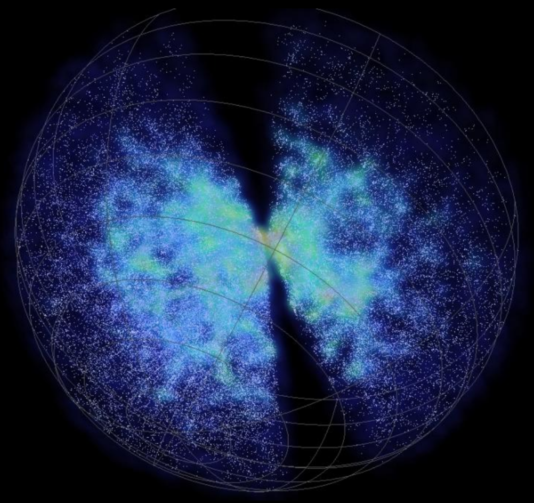
Scale-discretised wavelets form a Parseval (tight) frame:

$$A\|f\|^2 \leq \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) \left|\langle f, \mathcal{R}_\omega \Phi\rangle\right|^2 + \sum_{j=J_0}^{J} \int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \left|\langle f, \mathcal{R}_\rho \Psi^{(j)}\rangle\right|^2 \leq B\|f\|^2 ,$$

with $A = B = 1$, for any band-limited $f \in \mathrm{L}^2(\mathbb{S}^2)$, and where $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$.

(Adopt shorthand integral notation, although by appealing to sampling theorems and exact quadrature rules integrals may be replaced by finite sums.)

Galaxy distribution observed on the 3D ball



Credit: SDSS

Fourier-LAGuerre wavelets (flaglets) on the ball

- Fourier-Laguerre wavelet (flaglet) transform is given by the projection onto each wavelet (Leistedt & McEwen 2012; Lesitedt, McEwen, Kitching & Peiris 2015):

$$\boxed{W^{{}_s\Psi^{jj'}}(r,\rho) = \langle {}_sf,\, \mathcal{T}_{(r,\rho)}\, {}_s\Psi^{jj'} \rangle} = \int_{\mathbb{B}^3} \mathrm{d}^3\boldsymbol{r}\; {}_sf(\boldsymbol{r}) (\mathcal{T}_{(r,\rho)}\, {}_s\Psi^{jj'})^*(\boldsymbol{r}) \; .$$

projection

- Original function may be recovered exactly in practice from wavelet coefficients:

$${}_sf(\boldsymbol{r}) = \boxed{\sum_{j\,j'}} \boxed{\int_{\mathrm{SO}(3)} \mathrm{d}\varrho(\rho) \int_{\mathbb{R}^+} \mathrm{d}r\, W^{{}_s\Psi^{jj'}}(r,\rho) (\mathcal{T}_{(r,\rho)}\, {}_s\Psi^{jj'})(\boldsymbol{r}) \; .}$$

finite sum — wavelet contribution

- Define translation operator on positive real line $\mathbb{R}^+ = [0, \infty)$.

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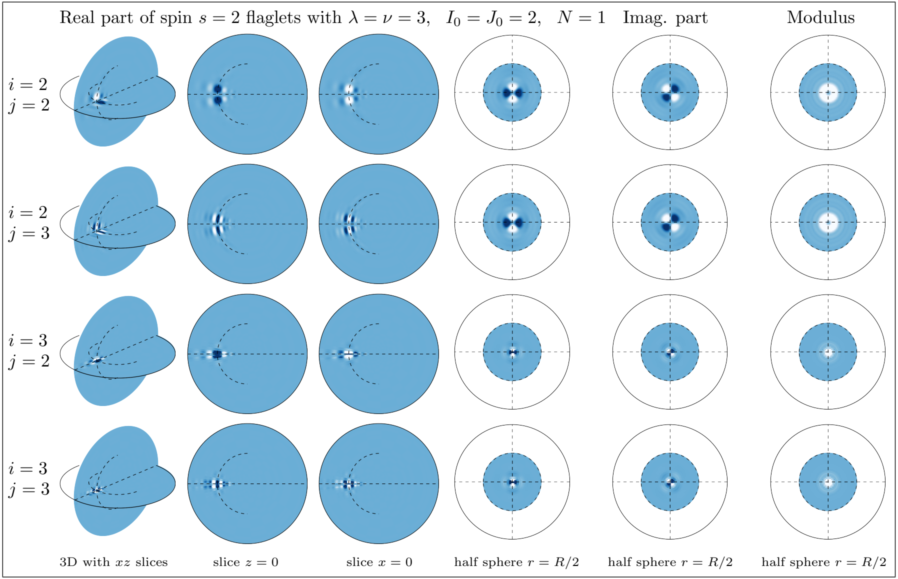

Outline

Sampling theory on the sphere $\mathbb{S}^2$

Exact and efficient spherical harmonic transforms

Equiangular sampling theorem on the sphere $\mathbb{S}^2$ (McEwen & Wiaux 2011)

Information content of a signal $f \in \mathrm{L}^2(\mathbb{S}^2)$ on the sphere $\mathbb{S}^2$, band-limited at L, can be captured in $\sim 2L^2$ equiangular samples.

Outline of proof: factoring of rotations, mapping of sphere $\mathbb{S}^2$ to torus $\mathbb{T}^2$, Fourier transform

- Previous canonical sampling theorem on the sphere based on the seminal work of Driscoll & Healy (1994)
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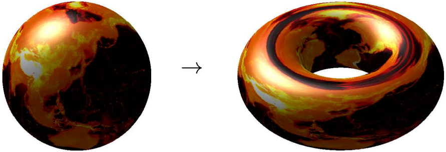

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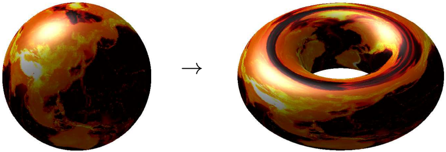

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Sampling theory on the rotation group $\mathrm{SO}(3)$

Exact and efficient Wigner transforms

- Wavelet coefficients for scale j live on the rotation group $\mathrm{SO}(3)$: $W^{{}_s\Psi^j} \in \mathrm{L}^2(\mathrm{SO}(3))$.
- Develop fast wavelet transforms by considering their (Wigner) harmonic representation.
- Signal on the rotation group $F \in \mathrm{L}^2(\mathrm{SO}(3))$ may expressed by Wigner decomposition:

$$F(\rho) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=-\ell}^{\ell} \frac{2\ell+1}{8\pi^2} F_{mn}^{\ell} \, D_{mn}^{\ell *}(\rho)$$

where Wigner coefficients given by usual projection onto basis functions:

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Fast Wigner transform
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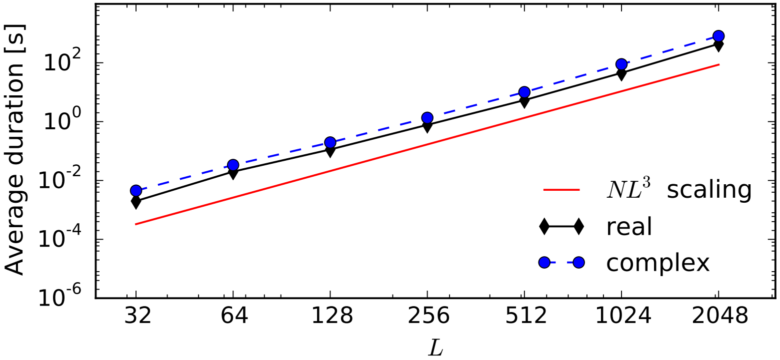

Figure: $N = 4$

Fast Wigner transform

Accuracy

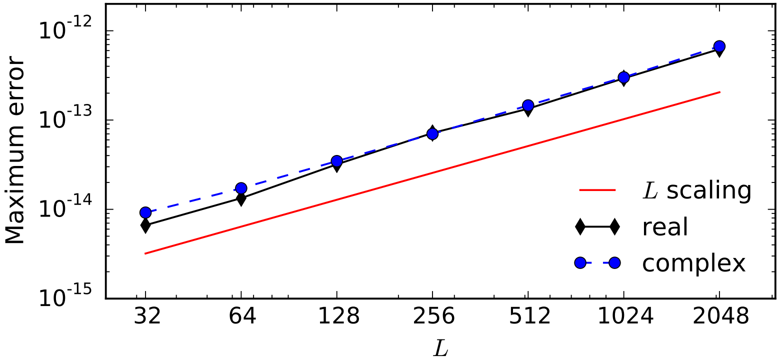

Figure: $N = 4$

Fast directional spin scale-discretised wavelet transform on the sphere

Exact and efficient computation via Wigner transforms

- Directional wavelet analysis can be posed as an inverse Wigner transform on $\mathrm{SO}(3)$:

$$\left(W^{{}_s\Psi^j}\right)^{\ell}_{mn} = \frac{8\pi^2}{2\ell+1}\, {}_s f_{\ell m}\, {}_s\Psi^{j*}_{\ell n} \,,$$

analysis

with

$$W^{{}_s\Psi^j}(\rho) = \sum_{\ell m n} \frac{2\ell+1}{8\pi^2} \left(W^{{}_s\Psi^j}\right)^{\ell}_{mn} D^{\ell *}_{mn}(\rho) \,.$$

- Directional wavelet synthesis can be posed as a forward Wigner transform on $\mathrm{SO}(3)$:

$${}_s f(\omega) = \sum_{j=0}^{J} \sum_{\ell m n} \frac{2\ell+1}{8\pi^2} \left(W^{{}_s\Psi^j}\right)^{\ell}_{mn} {}_s\Psi^{j}_{\ell n}\, {}_s Y_{\ell m}(\omega) \,,$$

synthesis

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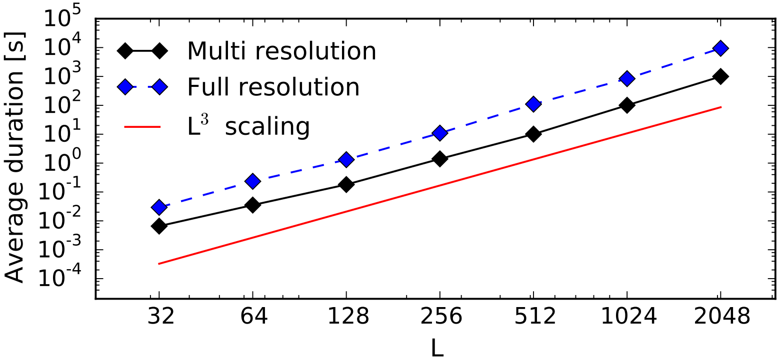


Figure: $N = 5$, $s = 2$

Fast directional spin scale-discretised wavelet transform on the sphere

Accuracy

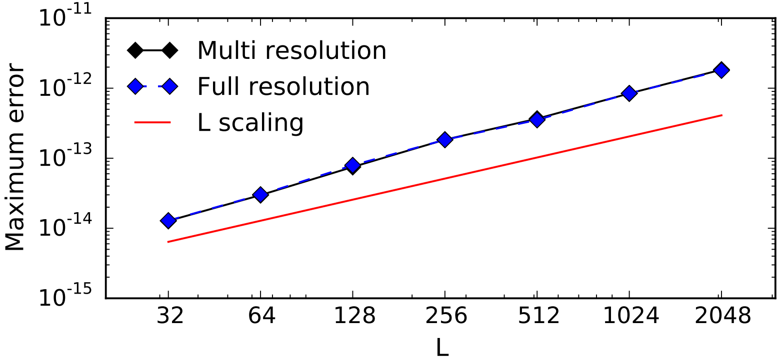


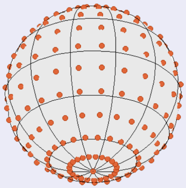
Figure: $N = 5,\ s = 2$

Sampling theory and harmonic transforms

Codes (www.jasonmcewen.org/codes.html)

SSHT code

http://www.spinsht.org



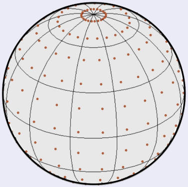
SSHT: Fast & exact spin spherical harmonic transforms

McEwen & Wiaux (2011)

- C, Matlab, Python
- Efficient sampling theorem on the sphere $\mathbb{S}^2$
- Fast algos

SO3 code

http://www.sothree.org



SO3: Fast & exact Wigner transforms

McEwen, Büttner, Leistedt, Peiris, Wiaux (2015)

- C, Matlab, Python
- Efficient sampling theorem on the rotation group $\mathrm{SO}(3)$
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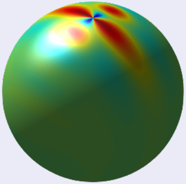
Spin scale-discretised wavelets on the sphere and ball

Codes (www.jasonmcewen.org/codes.html)

S2LET code

http://www.s2let.org



S2LET: Fast & exact wavelets on the sphere

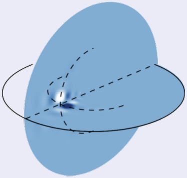
Leistedt, McEwen, Vandergheynst, Wiaux (2012)

McEwen, Leistedt, Büttner, Peiris, Wiaux (2015)

- C, Matlab, Python
- Supports directional, steerable, spin wavelets
- Fast algos

FLAGLET code

http://www.flaglets.org



FLAGLET: Fast & exact wavelets on the ball

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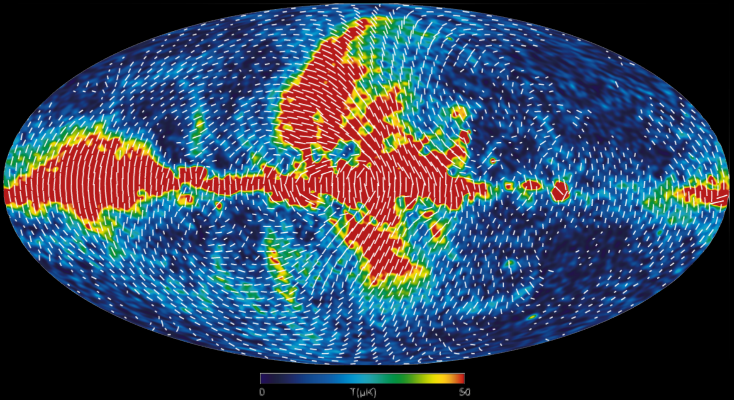
Outline

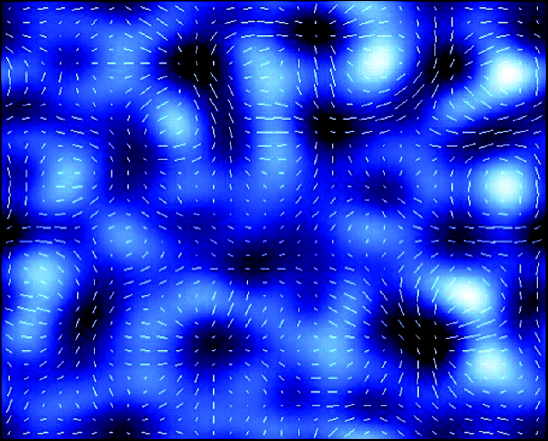
CMB polarization



WMAP K-band ${}_2P = Q + \mathrm{i}U$ map

[Credit: WMAP]

Cosmic shear



Cosmic shear ${}_2\gamma = \gamma_1 + \mathrm{i}\gamma_2$ map

[Credit: Ellis (2010)]

E- and B-modes

Full-sky

- Decompose ${}_{\pm 2}P$ into parity even and parity odd components:

$$\epsilon(\omega) = -\frac{1}{2}\left[\bar{\eth}^2 \, {}_2P(\omega) + \eth^2 \, {}_{-2}P(\omega)\right]$$ E-mode

$$\beta(\omega) = \frac{i}{2}\left[\bar{\eth}^2 \, {}_2P(\omega) - \eth^2 \, {}_{-2}P(\omega)\right]$$ B-mode

where $\bar{\eth}$ and $\eth$ are spin lowering and raising (differential) operators, respectively.

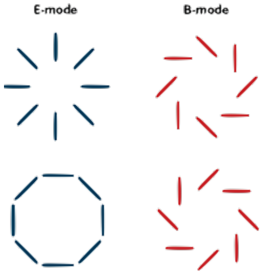


Figure: E-mode (even parity) and B-mode (odd parity) signals **[Credit:** http://www.skyandtelescope.com/**]**.

- Different physical processes exhibit different symmetries and thus behave differently under parity transformation.
- Can exploit this property to separate signals arising from different underlying physical mechanisms.
- Mapping E- and B-modes on the sky of great importance for forthcoming experiments.

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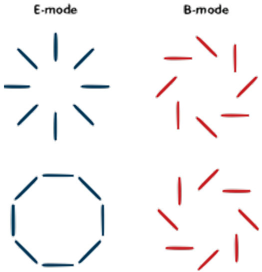

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E- and B-modes

Partial-sky

- On a manifold without boundary (*i.e.* full sky), a spin ± 2 signal can be decomposed uniquely into E- and B-modes.
- On a manifold with boundary (*i.e.* partial sky), decomposition not unique.
- Recovering E and B-modes from partial sky observations is challenging since mask leaks contamination.
- Pure and ambiguous modes (Lewis *et al.* 2002, Bunn *et al.* 2003, Smith 2006, Smith & Zaldarriaga 2007, Grain *et al.* 2007, Ferté *et al.* 2013).
  - E-modes: vanishing curl
  - B-modes: vanishing divergence
  - Pure E-modes: orthogonal to all B-modes
  - Pure B-modes: orthogonal to all E-modes
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E/B separation

Connections between spin and scalar wavelet coefficients

- Spin wavelet transform of ${}_{\pm 2}P = Q \pm \mathrm{i}U$ (observable):

$$\underbrace{W^{{}_2\Psi^j}_{{}_{\pm 2}P}(\rho) = \langle {}_{\pm 2}P,\, \mathcal{R}_\rho\, {}_{\pm 2}\Psi^j \rangle}_{\text{spin wavelet transform}} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega)\, {}_{\pm 2}P(\omega) (\mathcal{R}_\rho\, {}_{\pm 2}\Psi^j)^*(\omega) \; .$$

- Scalar wavelet transforms of E and B (non-observable):

$$\underbrace{W^{{}_0\Psi^j}_{\epsilon}(\rho) = \langle \epsilon,\, \mathcal{R}_\rho\, {}_0\Psi^j \rangle}_{\text{scalar wavelet transform}} \; ,$$

$$\underbrace{W^{{}_0\Psi^j}_{\beta}(\rho) = \langle \beta,\, \mathcal{R}_\rho\, {}_0\Psi^j \rangle}_{\text{scalar wavelet transform}} \; ,$$

where ${}_0\Psi^j \equiv \bar{\eth}^2\, {}_2\Psi^j$.

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$$W^{{}_0\Psi^j}_{\epsilon}(\rho) = -\mathrm{Re}\left[W^{{}_2\Psi^j}_{{}_{\pm 2}P}(\rho) \right] \quad \text{and} \quad W^{{}_0\Psi^j}_{\beta}(\rho) = \mp \mathrm{Im}\left[W^{{}_2\Psi^j}_{{}_{\pm 2}P}(\rho) \right] .$$

E/B separation

Connections between spin and scalar wavelet coefficients

- Spin wavelet transform of ${}_{\pm 2}P = Q \pm \mathrm{i}U$ (observable):

$$\boxed{W^{{}_2\Psi^j}_{{}_{\pm 2}P}(\rho) = \langle {}_{\pm 2}P,\, \mathcal{R}_\rho\, {}_{\pm 2}\Psi^j \rangle} = \int_{\mathbb{S}^2} \mathrm{d}\Omega(\omega) {}_{\pm 2}P(\omega) (\mathcal{R}_\rho\, {}_{\pm 2}\Psi^j)^*(\omega)\ .$$

spin wavelet transform

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scalar wavelet transform

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E/B separation

Exploiting wavelets

General approach to recover E/B signals using scale-discretised wavelets

1. Compute spin wavelet transform of ${}_{\pm2}P = Q + \mathrm{i}U$:

$${}_{\pm2}P(\omega) \xrightarrow[\text{S2LET}]{\text{Spin wavelet transform}} W^{{}_2\Psi^j}_{{}_{\pm2}P}(\rho)$$

2. Account for mask in wavelet domain (simultaneous harmonic and spatial localisation):

$$W^{{}_2\Psi^j}_{{}_{\pm2}P}(\rho) \xrightarrow{\text{Mitigate mask}} \bar{W}^{{}_2\Psi^j}_{{}_{\pm2}P}(\rho)$$

3. Construct E/B maps:

(a) $W^{{}_0\Psi^j}_{\epsilon}(\rho) = -\mathrm{Re}\left[\bar{W}^{{}_2\Psi^j}_{{}_{\pm2}P}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \epsilon(\omega)$

(b) $W^{{}_0\Psi^j}_{\beta}(\rho) = \mp\mathrm{Im}\left[\bar{W}^{{}_2\Psi^j}_{{}_{\pm2}P}(\rho)\right] \xrightarrow[\text{S2LET}]{\text{Inverse scalar wavelet transform}} \beta(\omega)$

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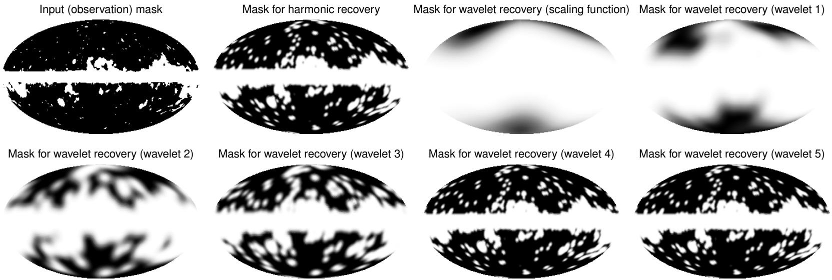
E/B separation
Scale-dependent masking



E/B separation

Pure mode wavelet estimator

- Consider masked Stokes parameters:

$$ {}_0M = M, \quad {}_{\pm1}M = \eth_\pm M, \quad {}_{\pm2}M = \eth_\pm^2 M, $$

spin adjusted masks

$$ {}_{\pm2}\widetilde{P} = {}_0M\,{}_{\pm2}P, \quad {}_{\pm1}\widetilde{P} = {}_{\mp1}M\,{}_{\pm2}P, \quad {}_{\pm0}\widetilde{P} = {}_{\mp2}M\,{}_{\pm2}P. $$

masked Stokes parameters

where $\eth_\pm = \{\, \eth \text{ if } + \,,\ \bar{\eth} \text{ if } - \,\}$.

- Pure wavelet estimators (Leistedt, McEwen, Büttner, Peiris 2016):

$$ \widehat{W}_{\epsilon}^{0\Psi^j}(\rho) = -\,\mathrm{Re}\left[W^{\pm2\Upsilon^j}_{\pm2\widetilde{P}}(\rho) + 2W^{\pm1\Upsilon^j}_{\pm1\widetilde{P}}(\rho) + W^{0\Upsilon^j}_{0\widetilde{P}}(\rho) \right], $$

pure E

$$ \widehat{W}_{\beta}^{0\Psi^j}(\rho) = \mp\,\mathrm{Im}\left[W^{\pm2\Upsilon^j}_{\pm2\widetilde{P}}(\rho) + 2W^{\pm1\Upsilon^j}_{\pm1\widetilde{P}}(\rho) + W^{0\Upsilon^j}_{0\widetilde{P}}(\rho) \right], $$

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where ${}_{\pm s}\Upsilon^j = \eth_\pm^s({}_0\Psi^j)$ are spin adjusted wavelets and assuming the Dirichlet and Neumann boundary conditions, *i.e.* that the mask and its derivative vanish at the boundaries.

E/B separation

Pure mode wavelet estimator

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where $\eth_{\pm} = \{ \, \eth \text{ if } + \, , \, \bar{\eth} \text{ if } - \, \}$.

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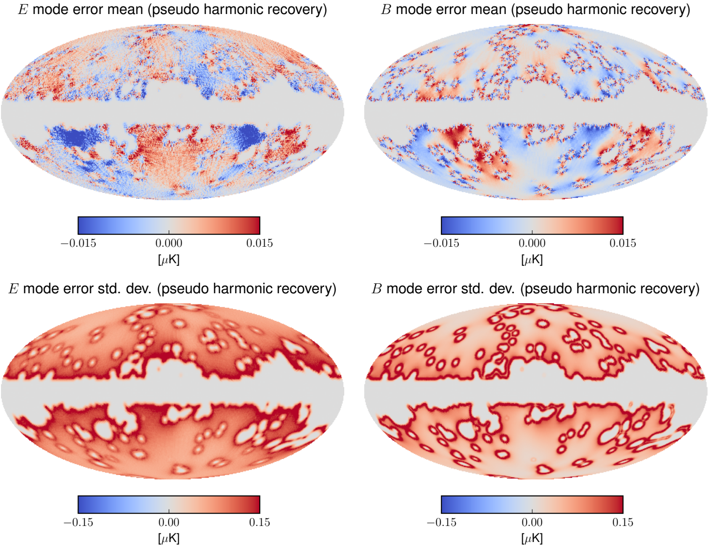
- Correction terms require spin ±1 wavelet transforms.

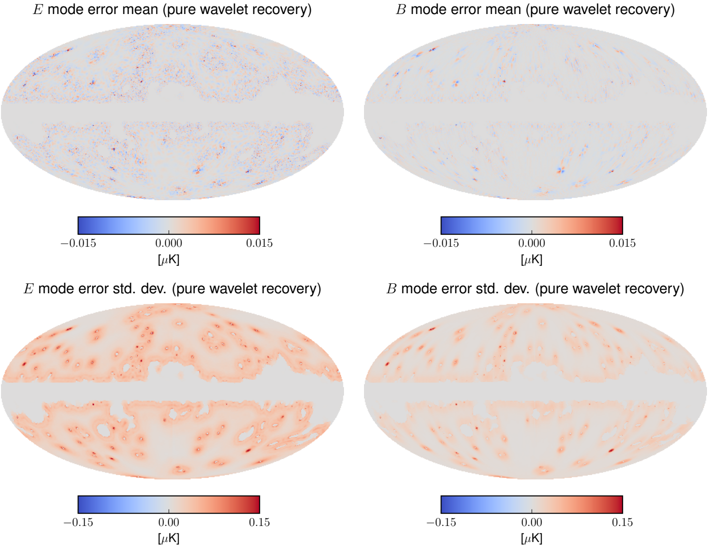
E/B separation

Results: pseudo harmonic approach



E/B separation

Results: pure wavelet approach



Summary

Spin scale-discretised wavelets on the sphere $\mathbb{S}^2$ and ball $\mathrm{SO}(3)$ are powerful tools for studying CMB and weak gravitational lensing and beyond (e.g. diffusion MRI).

- Exact forward (analysis) and inverse (synthesis) transforms in theory and practice.
- Probe directional structure.
- Framework applies to signals of any spin.
- Excellent localisation properties
  (localisation of Gaussian random fields).
- Parseval frame.
- Fast algorithms to scale to big-data
  (leveraging exact and efficient harmonic transforms on $\mathbb{S}^2$ and $\mathrm{SO}(3)$).
- Elegant and practical connection between spin and scalar wavelet transforms
  (*e.g.* for E/B separation).