Morphological Analysis of cosmological random fields using Minkowski Functionals

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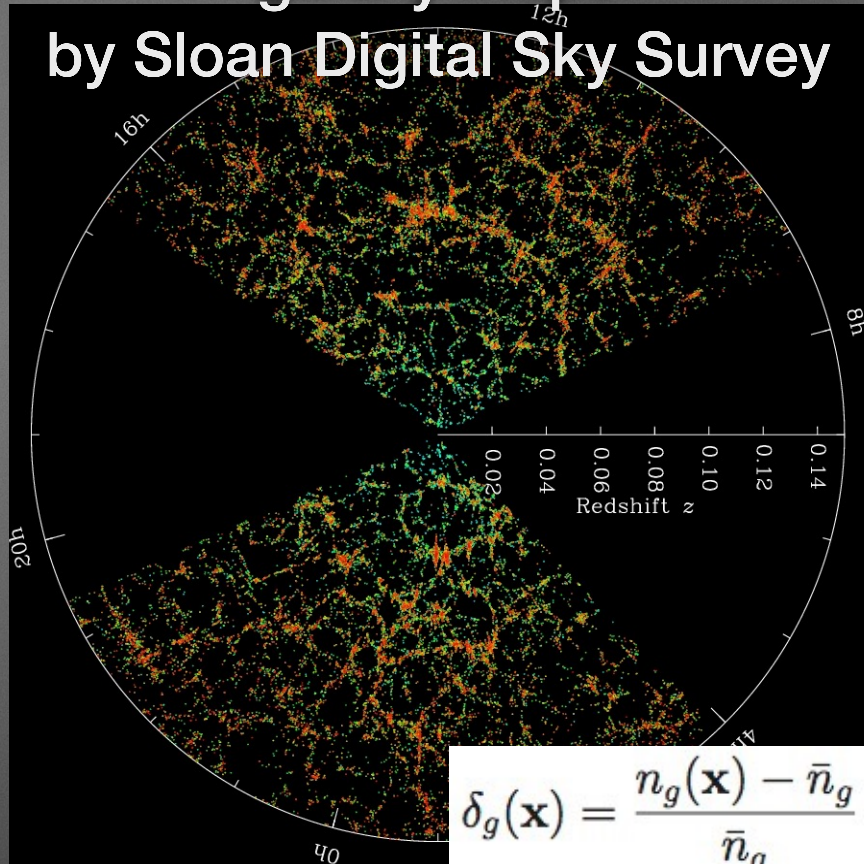
Outline

- Introduction of Minkowski Functionals
- Application of Minkowski Functionals to CMB temperature and B-mode polarization map
- Tensor Minkowski Functionals as anisotropic measures
- Summary

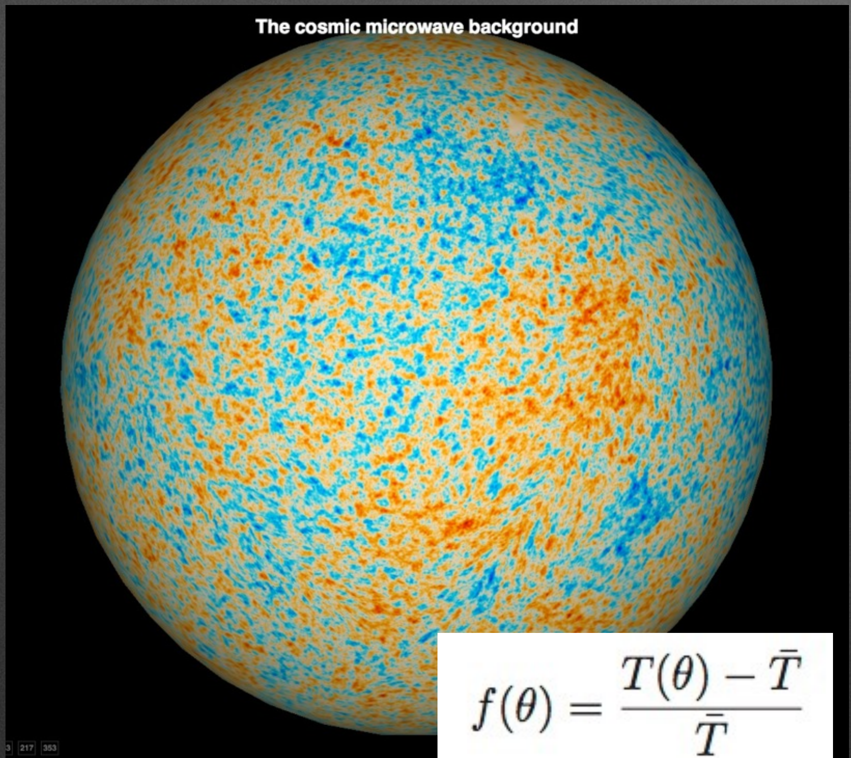
Observational data

galaxy maps
by Sloan Digital Sky Survey



$$\delta_g(\mathbf{x}) = \frac{n_g(\mathbf{x}) - \bar{n}_g}{\bar{n}_g}$$

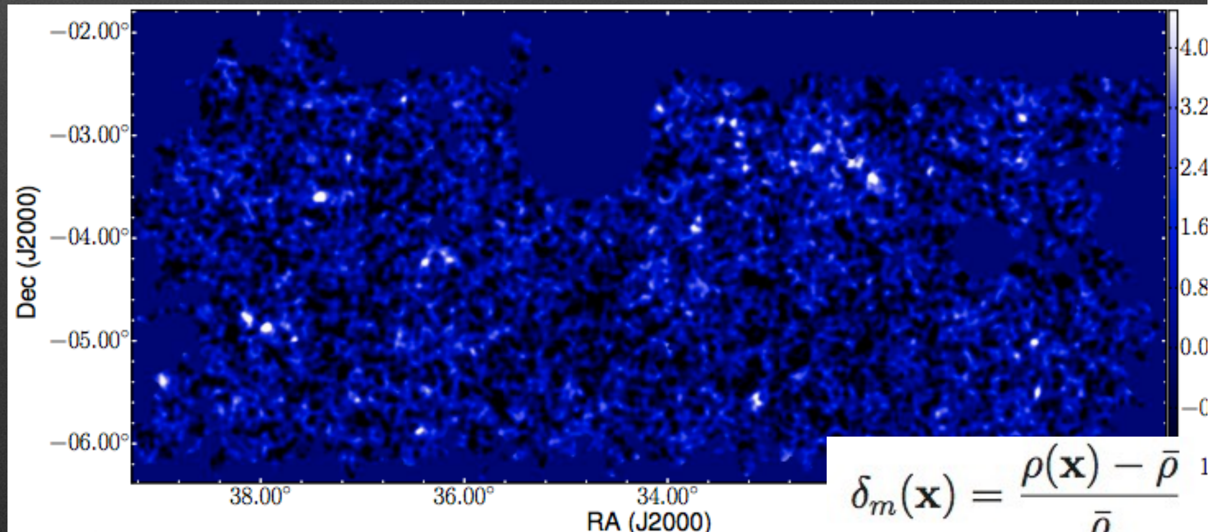
cosmic microwave background
observed by Planck satellite



$$f(\theta) = \frac{T(\theta) - \bar{T}}{\bar{T}}$$

temperature anisotropy

Weak lensing mass map by Hyper-Suprime Cam



$$\delta_m(\mathbf{x}) = \frac{\rho(\mathbf{x}) - \bar{\rho}}{\bar{\rho}}$$

Oguri et al. 2017

Gaussian random field

- In a Gaussian random field *G(x)*, the joint probability function of the field values $\alpha_i = G(x_i)$ with $i=1,\ldots,n$ becomes a multivariate Gaussian distribution

$$\mathcal{P}_n(\alpha_1,\ldots,\alpha_n)d\alpha_1\ldots d\alpha_n = [(2\pi)^n \det M]^{-1/2} \exp\left[-\frac{1}{2}\alpha_i M_{ij}^{-1}\alpha_j\right] d\alpha_1\ldots d\alpha_n$$

$M_{ij} = \langle \alpha_1 \alpha_j \rangle = \xi(x_{ij})$ covariance matrix

- A Gaussian random field with vanishing mean value is completely determined by the two-point correlation functions

Fourier space

- After Fourier-transform of the field *G(x)*, each Fourier mode is given by the amplitude and the phase

Fourier phase

$$\widetilde{G}(\mathbf{k}) = |\widetilde{G}(\mathbf{k})|\, e^{i\theta(\mathbf{k})}$$

Fourier amplitude

- In a Gaussian random field, each Fourier modes are statistically independent, and have random phases and the moduli |G(k)| follows Rayleigh distribution

$$\mathcal{P}[|\widetilde{G}(\mathbf{k})|, \theta(\mathbf{k})] = \exp\left[-\frac{|\widetilde{G}(\mathbf{k})|^2}{P(k)}\right] \frac{2\,|\widetilde{G}(\mathbf{k})|^2}{P(k)}\, d|\widetilde{G}(\mathbf{k})|\, \frac{d\theta(\mathbf{k})}{2\pi}$$

- Power spectrum *P(k)* completely describes the statistics of the random-Gaussian fields

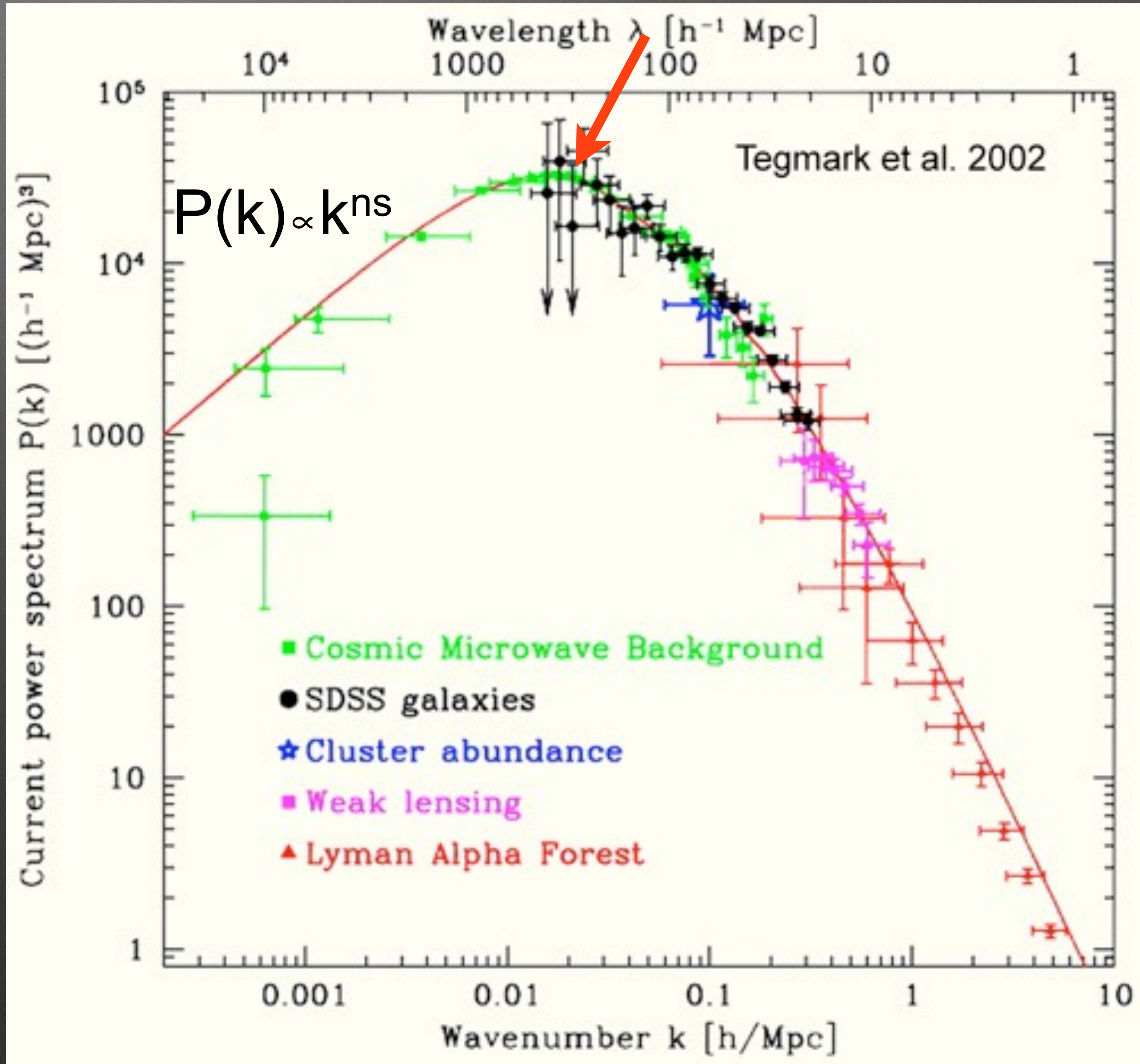
Power spectrum of cosmic matter density field

$P(k)=\langle|\delta_k|^2\rangle$

Square of the amplitude of the matter density fluctuation as a function of wavenumber of k

P(k) is measured from different probes such as CMB, galaxy distributions, weak lensing

horizon scale at matter-radiation equality time



Statistics for Non-Gaussian fields

- Power spectrum (or 2-point correlation functions) cannot describe non-Gaussian properties
- Higher-order statistics beyond 2-point statistics are necessary
  - e.g., bispectrum (3-point), trispectrum (4-point)
- Since non-Gaussianity has infinite freedom in general, there is no single statistic to fully characterize non-Gaussian properties

Minkowski Functionals (MFs)

- MFs characterize morphological properties
- In 2D space, there exist three MFs

V_0 Area

V_1 circumference

V_2 Euler characteristic

Hermann Minkowski
(1864-1909, Germay)

e.g., UK

V_0:0.00048 ($A/4\pi R^2$,A=244820km^2)

V_1:2.0 (l/R,l=12800km)

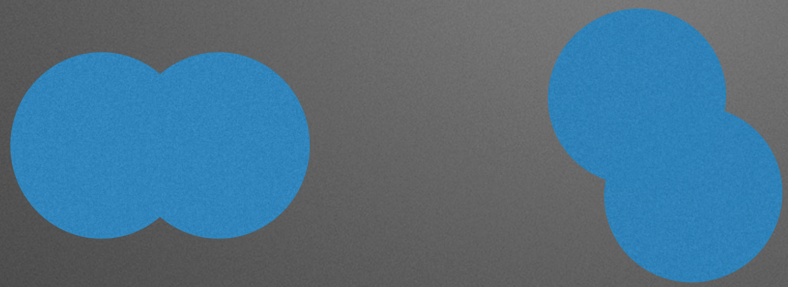
V_2:33=34−1 （# of islands – # of lakes [>100km^2])

Note: Values of MFs depend on a smoothing scale

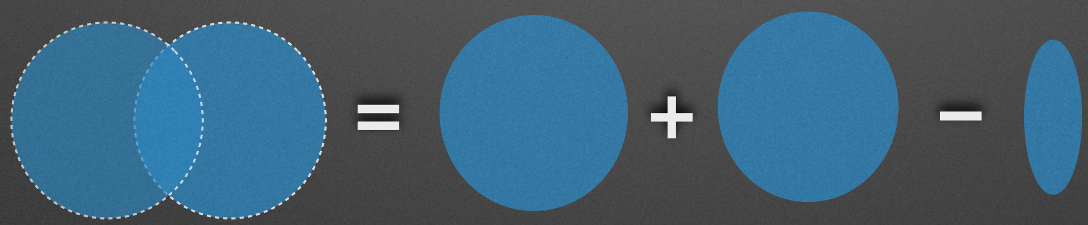
Conditions that MFs satisfy

1. Motion Invariance: $V_k(K)=V_k(gK)$ (g=rotation+transfer)

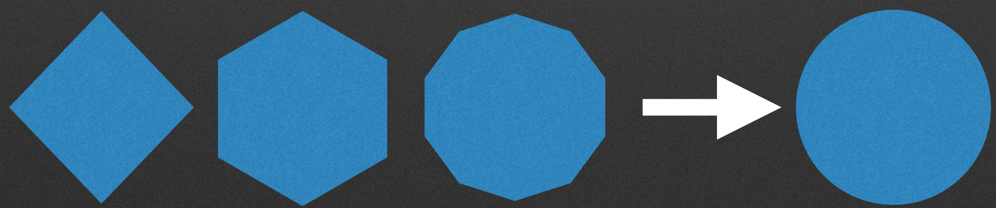
K : convex bodies



2. Additivity: $V_k(K_1 \cup K_2)=V_k(K_1)+V_k(K_2)-V_k(K_1 \cap K_2)$



3. Continuity: $V_k(K')\to V_k(K)$ as $K'\to K$



Hadwiger's theorem (1957)

- In d-dimensional space, there exists **d+1 number** of MFs V_k (k=0,1,…, d)

d	1	2	3
V_0	length	area	volume
V_1	χ	circumference	surface area
V_2	–	χ	total mean curvature
V_3	–	–	χ

χ: Euler characteristic

- Any morphological descriptors satisfying motion-invariant, additive, and continuous conditions is a linear combination of MFs

$$M = \sum_{k=0}^{d} a_k V_k^{(d)} \quad \text{with numbers } a_k$$

Hugo Hadwiger (1908-1981)
Swiss mathematician

Steiner’s formula

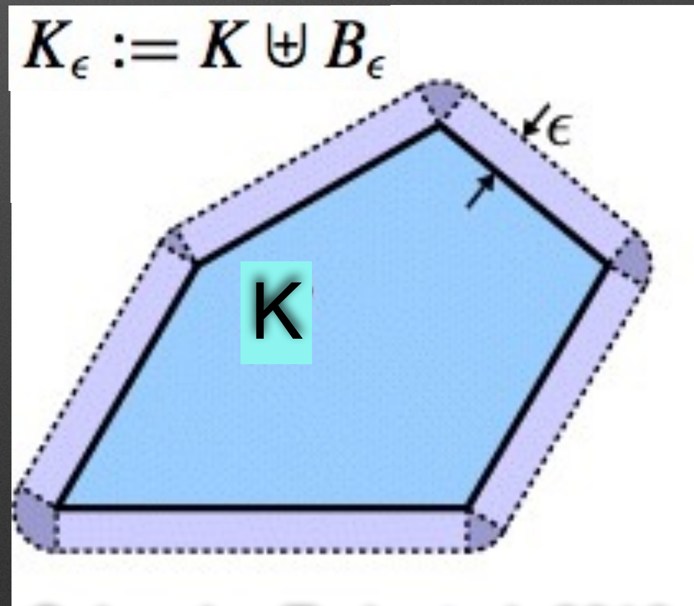
- For a given body *K*, the MFs of the parallel body K_ε at a distance ε from *K* are a polynomial ε with coefficients proportional to the MFs of *K*

$$V_i(K_\epsilon) = \sum_{j=i}^{d} \binom{d-i}{j-i} V_j(K)\epsilon^{j-i}$$

Jakob Steiner (1796-1863)
Swiss mathematician

e.g., 2D body

$$V_0(K_\epsilon) = V_0(K) + 2V_1(K)\epsilon + V_2(K)\epsilon^2$$
$$V_1(K_\epsilon) = V_1(K) + V_2(K)\epsilon$$
$$V_2(K_\epsilon) = V_2(K)$$



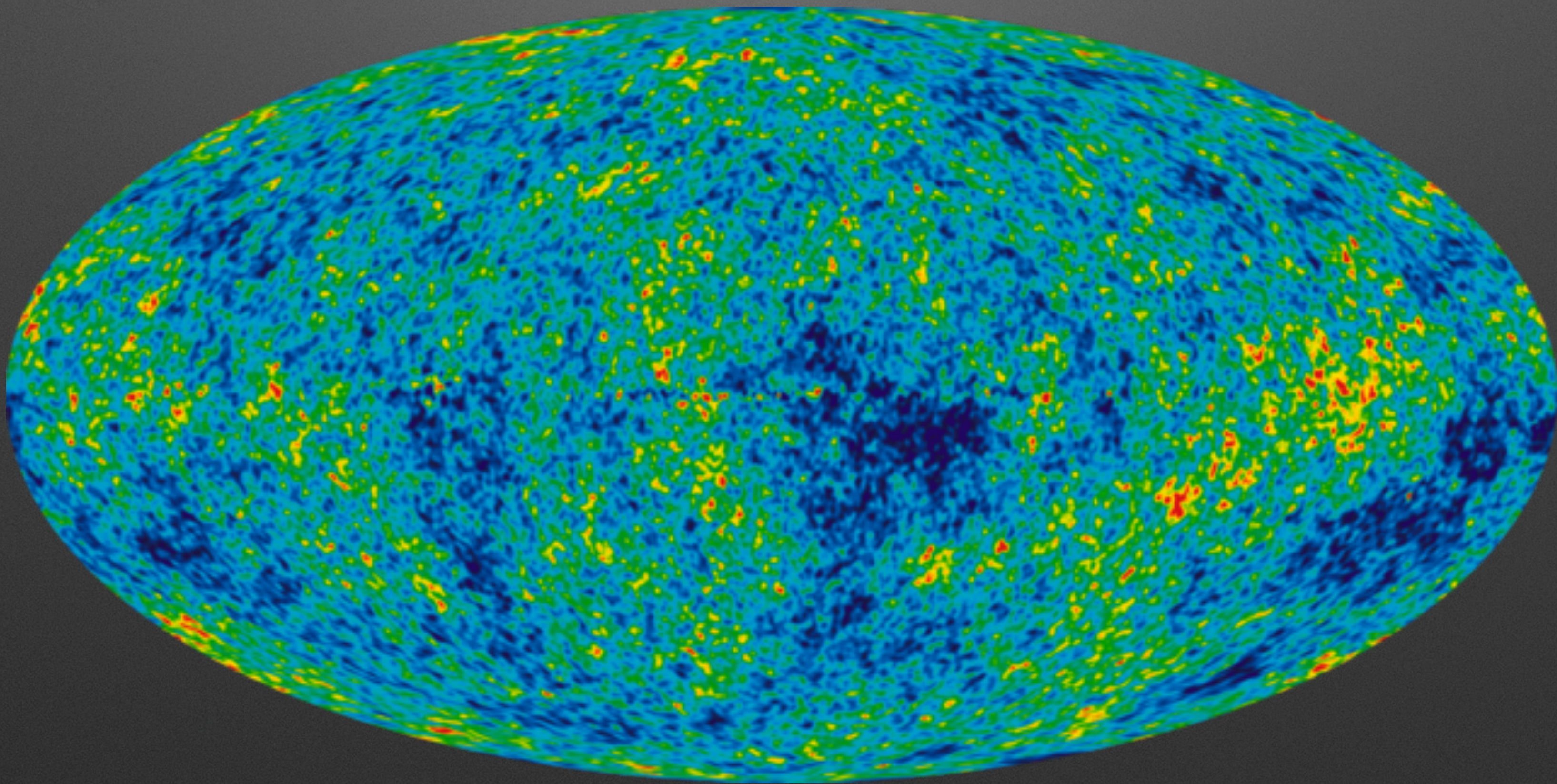
Schroder-Turk et al. 2010

V_0 Area, V_1 circumference, V_2 Euler characteristic

MFs for a scalar field

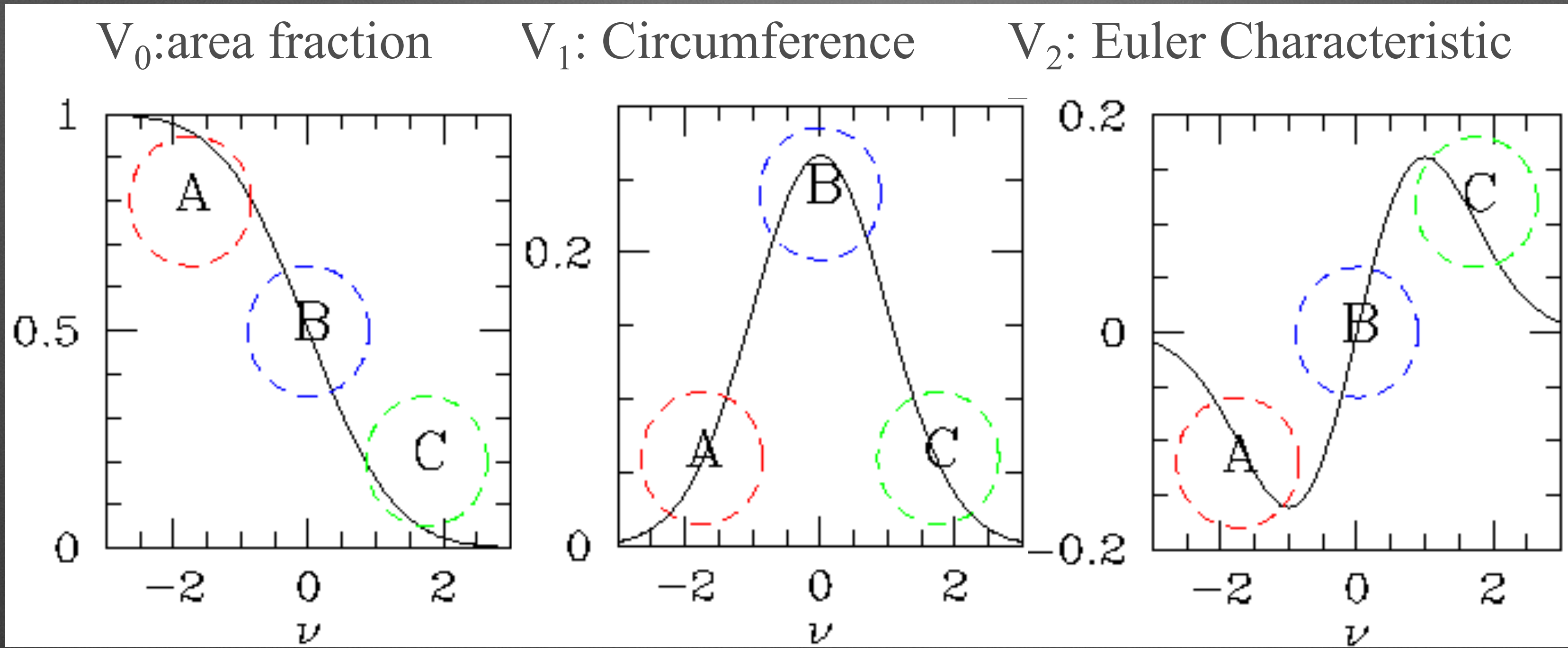
- Let's consider a smooth scalar field *f(θ)* on the sphere (e.g., CMB)
- MFs are measured in the excursion set over a given threshold *ν*

$$K_\nu = (\theta \in \mathcal{S}^2 | f(\theta) > \nu) \quad \nu = f/\sigma_0 \quad \sigma_0 = \langle f^2 \rangle^{1/2}$$



(c) WMAP team (data), T. Matsubara (movie)

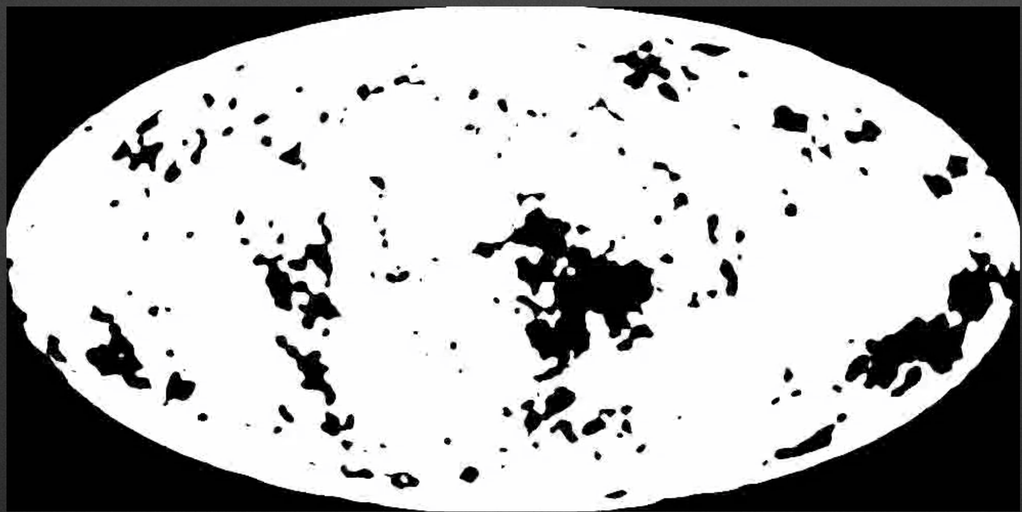
MFs as a function of threshold ν



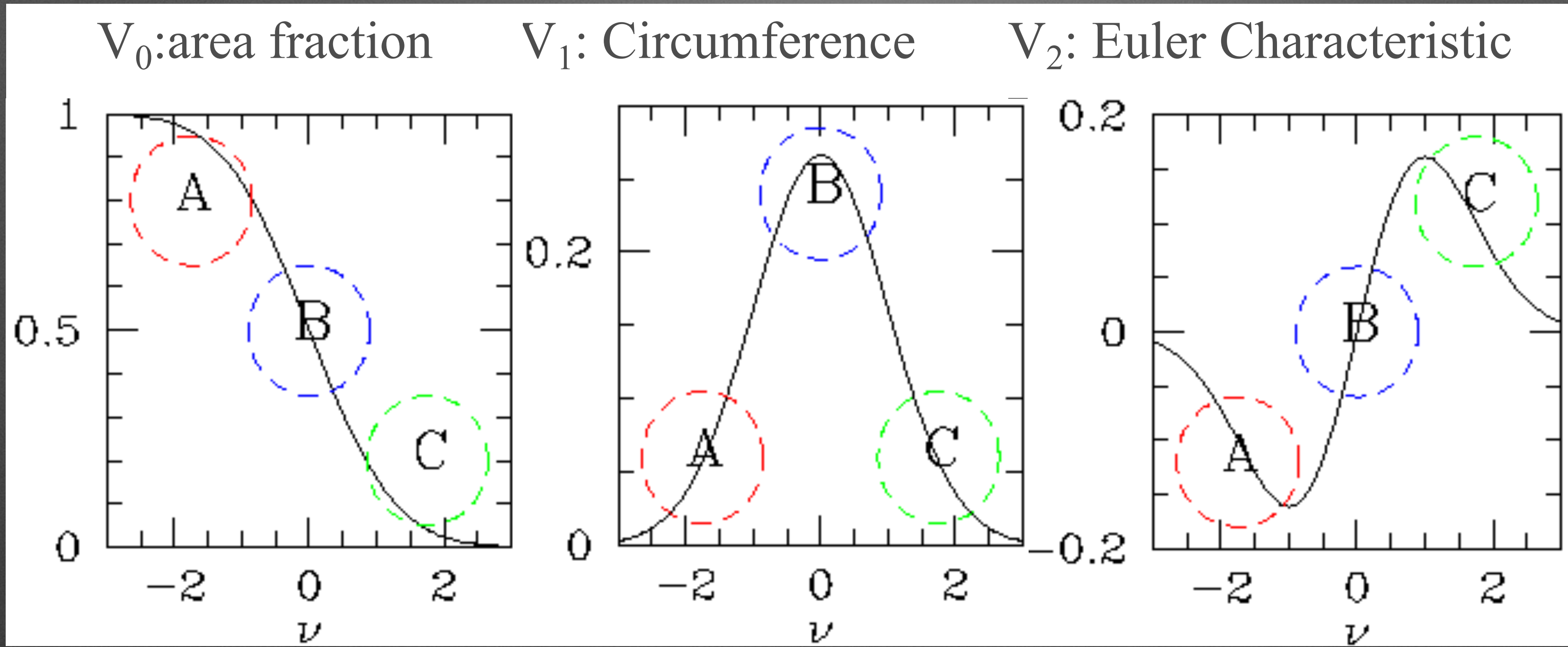
$$\nu = f/\sigma_0$$

$$\sigma_0 = \langle f^2 \rangle^{1/2}$$

A



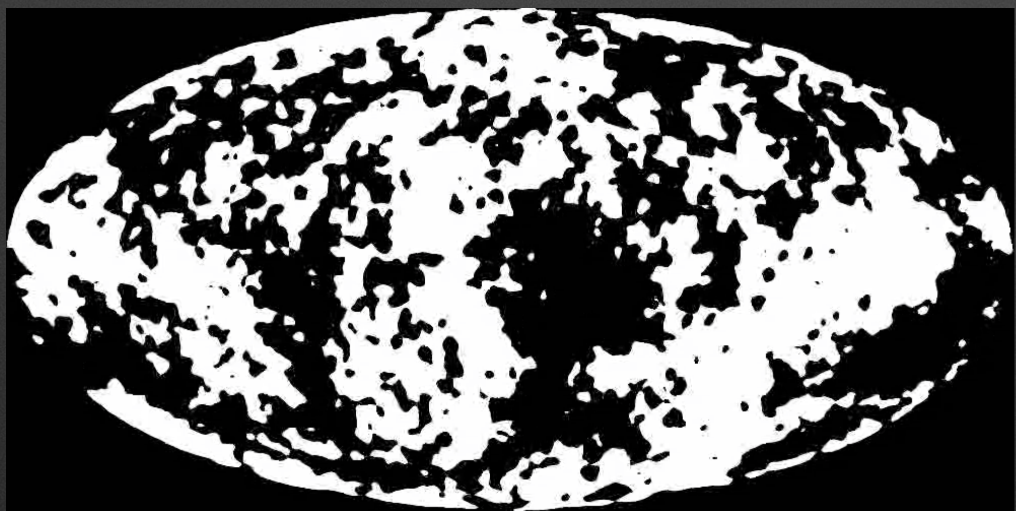
MFs as a function of threshold *ν*



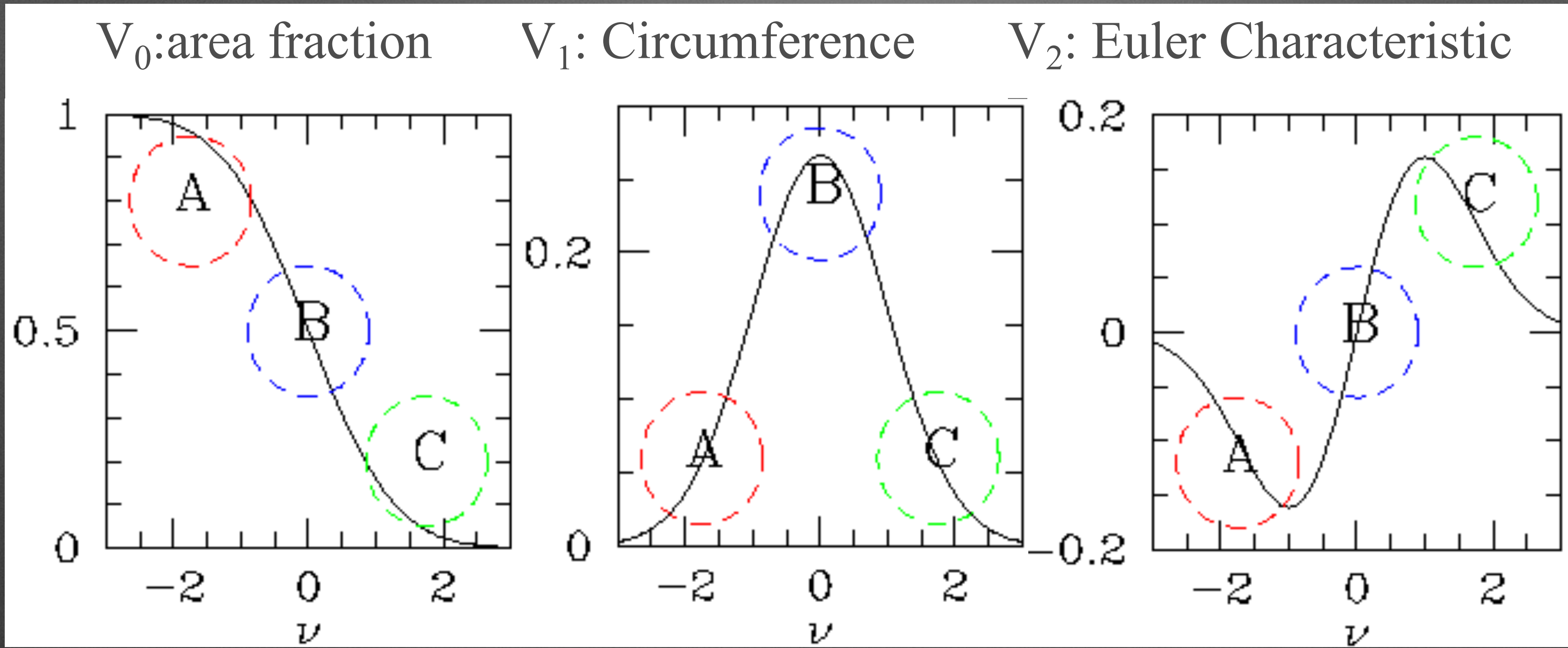
$$\nu = f/\sigma_0$$

$$\sigma_0 = \langle f^2 \rangle^{1/2}$$

B



MFs as a function of threshold ν



$\nu = f/\sigma_0$

$\sigma_0 = \langle f^2 \rangle^{1/2}$

C

Analytical formula of MFs in Gaussian random field

- Tomita's formula (Tomita 1986)

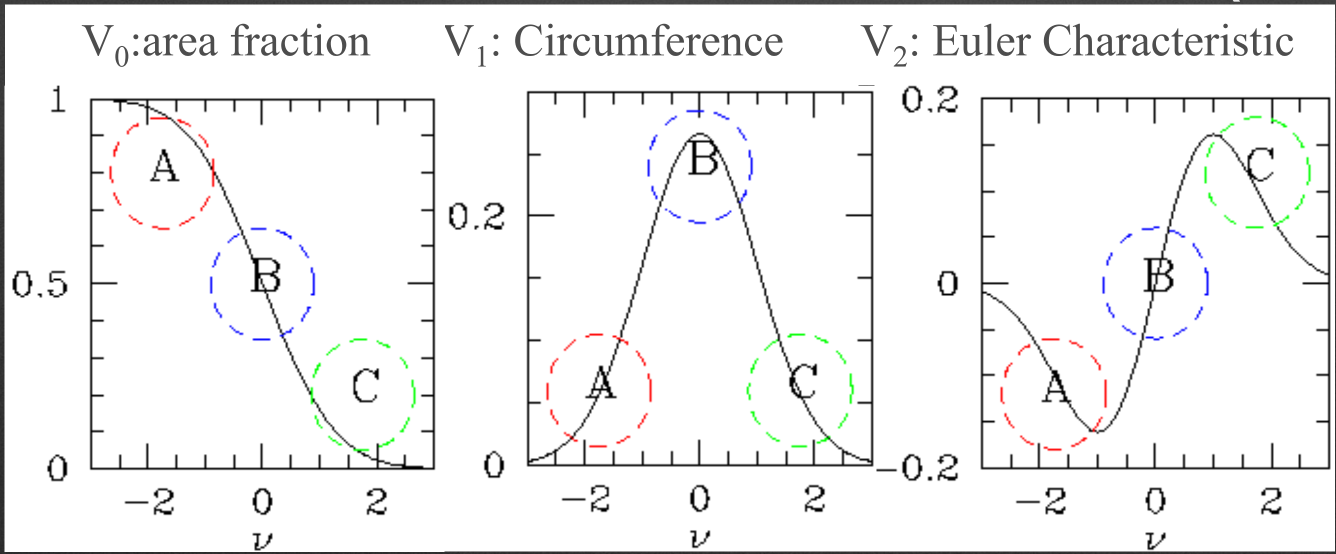
$$V_k^{(d)}(\nu) = \frac{1}{(2\pi)^{(k+1)/2}} \frac{\omega_d}{\omega_{d-k}\omega_k} \left(\frac{\sigma_1}{\sqrt{d}\sigma_0}\right)^k e^{-\nu^2/2} H_{k-1}(\nu)$$

Hermite polynomials

where

$$\nu = f/\sigma_0 \qquad \sigma_0 = \langle f^2 \rangle^{1/2} \qquad \sigma_1 = -\langle f \cdot \nabla^2 f \rangle^{1/2}$$

ω_k: volume of the unit ball in k-dimension ($\omega_0=1$, $\omega_1=2$, $\omega_2=\pi$)



Matsubara's formulae

Takahiko Matsubara
(professor@KEK)

- T. Matsubara 2003 derive the perturbative formula of MFs using multidimensional Edgeworth expansion

$$V_k(\nu) = V_k^{(\mathrm{G})}(\nu) + A_k e^{-\nu^2/2}[v_k^{(1)}(\nu)\sigma_0$$

Gaussian term leading-order perturbative term

- Leaing-order NG term is determined by three "skewness parameters"

$$v_k^{(1)}(\nu) = \frac{S}{6}H_{k+2}(\nu) - \frac{S_{\mathrm{I}}}{2}H_k(\nu) - \frac{S_{\mathrm{II}}}{2}H_{k-2}(\nu)$$

$$S \equiv \frac{\langle f^3 \rangle}{\sigma_0^4}, \quad S_{\mathrm{I}} \equiv \frac{f^2\langle \nabla^2 f \rangle}{\sigma_0^2 \sigma_1^2}, \quad S_{\mathrm{II}} \equiv \frac{2\langle |\nabla f|^2 \nabla^2 f \rangle}{\sigma_1^4}$$

- The skewness parameters are the sum of bispectra with different configuration weight

2nd-order perturbation of MFs

- Matsubara 2010 derive the 2nd-order perturbations of MFs, which depend on 4 kurtosis parameters

$$V_k(\nu) = V_k^{(\mathrm{G})}(\nu) + A_k e^{-\nu^2/2}[v_k^{(1)}(\nu)\sigma_0 + v_k^{(2)}(\nu)\sigma_0^2]$$

$$\begin{aligned}
v_0^{(2)}(\nu) &= \frac{S^2}{72}H_5(\nu) + \frac{K}{24}H_3(\nu), \\
v_1^{(2)}(\nu) &= \frac{S^2}{72}H_6(\nu) + \frac{K - SS_{\mathrm{I}}}{24}H_4(\nu) - \frac{1}{12}\left(K_{\mathrm{I}} + \frac{3}{8}S_{\mathrm{I}}^2\right)H_2(\nu) - \frac{K_{\mathrm{III}}}{8}, \\
v_2^{(2)}(\nu) &= \frac{S^2}{72}H_7(\nu) + \frac{K - 2SS_{\mathrm{I}}}{24}H_5(\nu) - \frac{1}{6}\left(K_{\mathrm{I}} + \frac{1}{2}SS_{\mathrm{II}}\right)H_3(\nu) - \frac{1}{2}\left(K_{\mathrm{II}} + \frac{1}{2}S_{\mathrm{I}}S_{\mathrm{II}}\right)H_1(\nu)
\end{aligned}$$

- Kurtosis parameters are the sum of trispectrum with different configuration weights

$$K \equiv \frac{\langle f^4\rangle_c}{\sigma_0^4}, \quad K_{\mathrm{I}} \equiv \frac{\langle(\nabla^2 f)f^3\rangle_c}{\sigma_0^4\sigma_1^2}, \quad K_{\mathrm{II}} \equiv \frac{2\langle f|\nabla f|^2\nabla^2 f\rangle_c + \langle|\nabla f|^4\rangle_c}{\sigma_0^2\sigma_1^4}, \quad K_{\mathrm{III}} \equiv \frac{\langle|\nabla f|^4\rangle_c}{2\sigma_0^2\sigma_1^4},$$

Primordial Non-Gaussianity

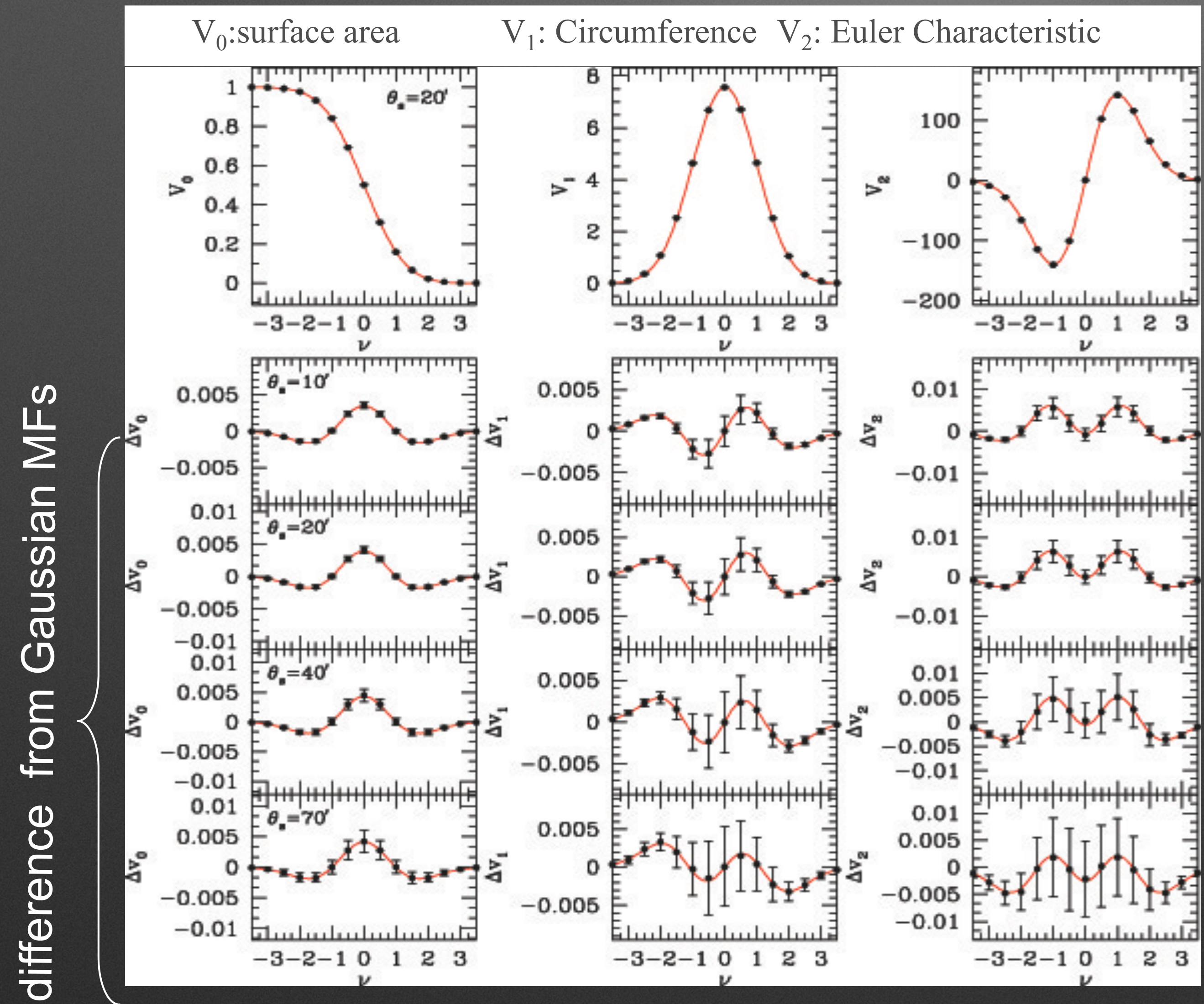
- Local-type non-Gaussianity in primordial perturbation (e.g., Komatsu, Spergel 2001)

$$\Phi = \phi + f_{\rm NL}(\phi^2 - \langle\phi^2\rangle) + g_{\rm NL}\phi^3 \ldots$$

Φ: auxiliary Gaussian variable

- Primoridal non-Gaussianity is a useful probe to differentiate models in early Universe
  - The simple model of inflation predicts f_{NL}~O(1), while other models can generate larger non-Gaussianity f_{NL}~O(1-100))

Perturbative formula vs Simulations with primordial NG



- · mock simulations
- — Perturbative formulae

f_{NL}=100

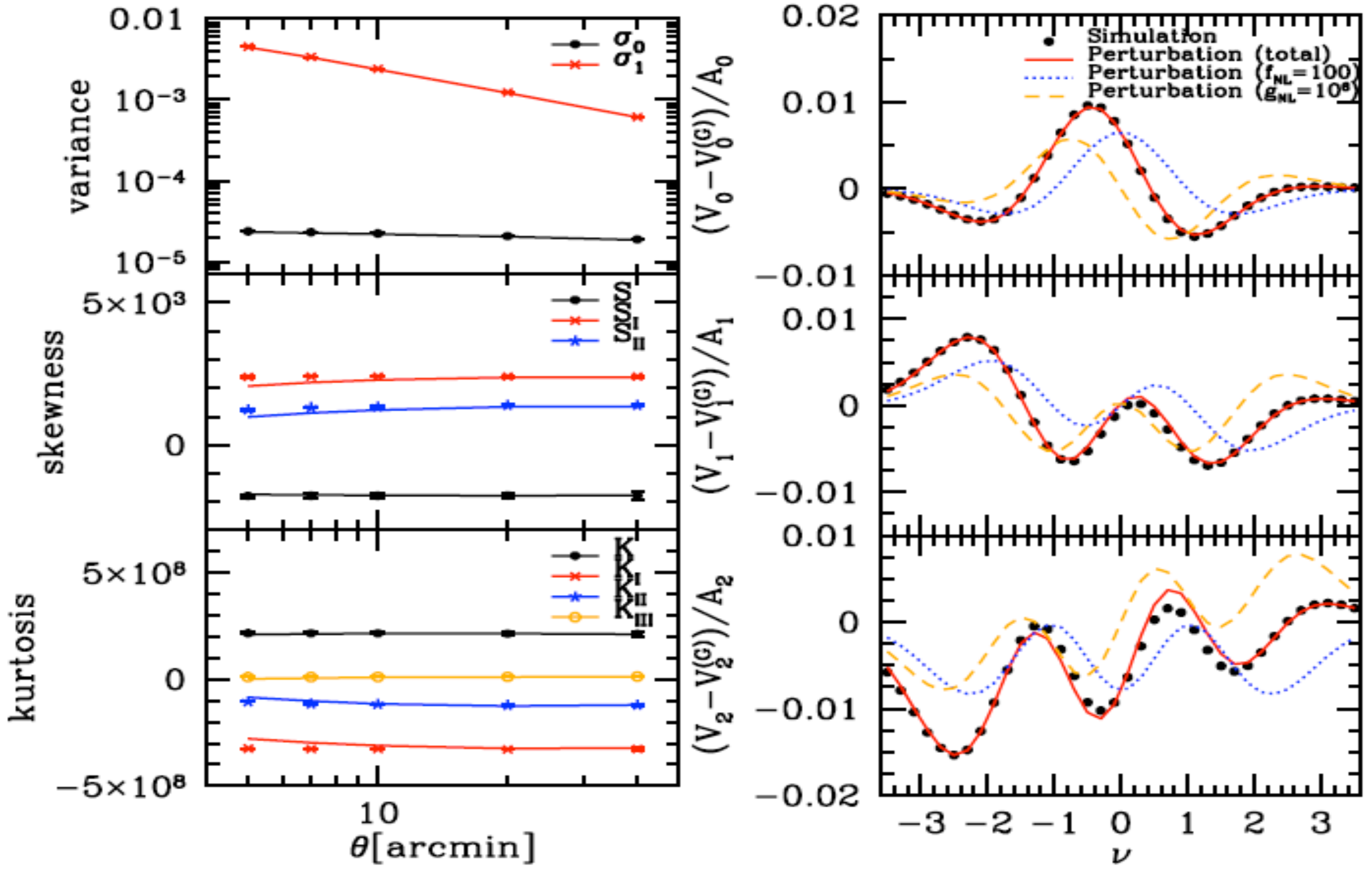
Error: dispersion of MFs among 1000 mock simulations

The agreement between simulations and perturbative formulae is perfect

CH, Matsubara, Coles et al. 2007

g_{NL}-type NG effect on MFs

2nd-order perturbation of MFs due to g_{NL}-type NG is also very good agreement with the simulation results



CH, Matsubara 2011

MFs

- **MFs depend on all-orders of polyspectra** (bispectra, trispectra, ….)
  - MFs and polyspectra play a complimentary role in the analysis of non-Gaussianity with each other
- **Computation of MFs are much faster** than the full calculation of bispectra/trispectra
- **MFs are model-independent statistics** and may be able to detect unexpected NG (e.g., unknown systematics), while the commonly used cubic statistics is optimal but model-dependent

Constraints on primordial NG from CMB data

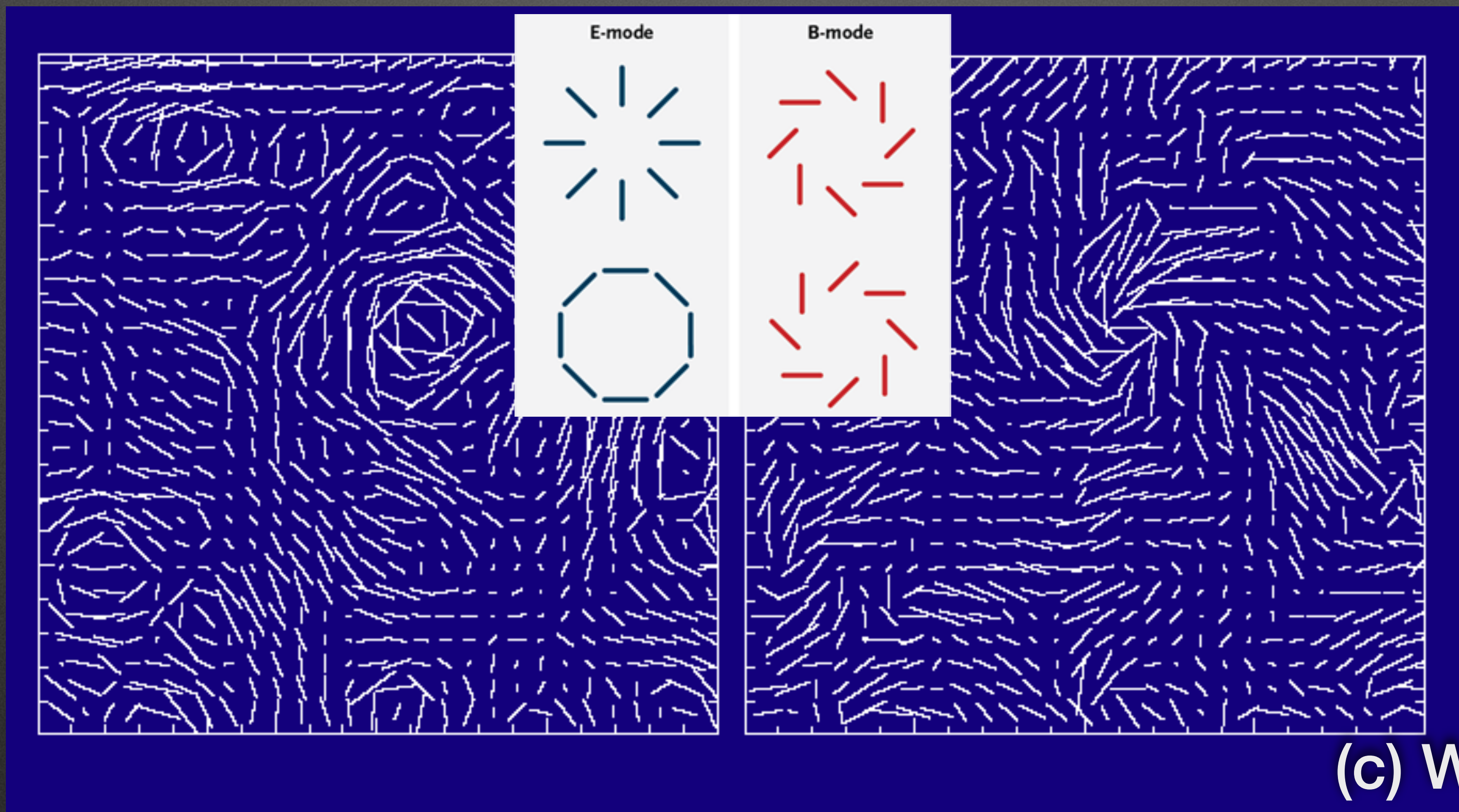
- WMAP
  - optimal cubic estimators: $f_{NL}=32\pm21$ (Komatsu et al. 2011)
  - MFs: $f_{NL}=20\pm42$, $g_{NL}=(-1.9\pm6.4)\times10^5$ (C.H., Matsubara 2012)
- Planck (Planck collaboration 2015)
  - optimal estimators: $f_{NL}=0.8\pm5.0$, $g_{NL}=(-9.0\pm7.7)\times10^4$
  - MFs: $f_{NL}=3\pm12$, $g_{NL}=(-8\pm13)\times10^4$

all error values are 1σ

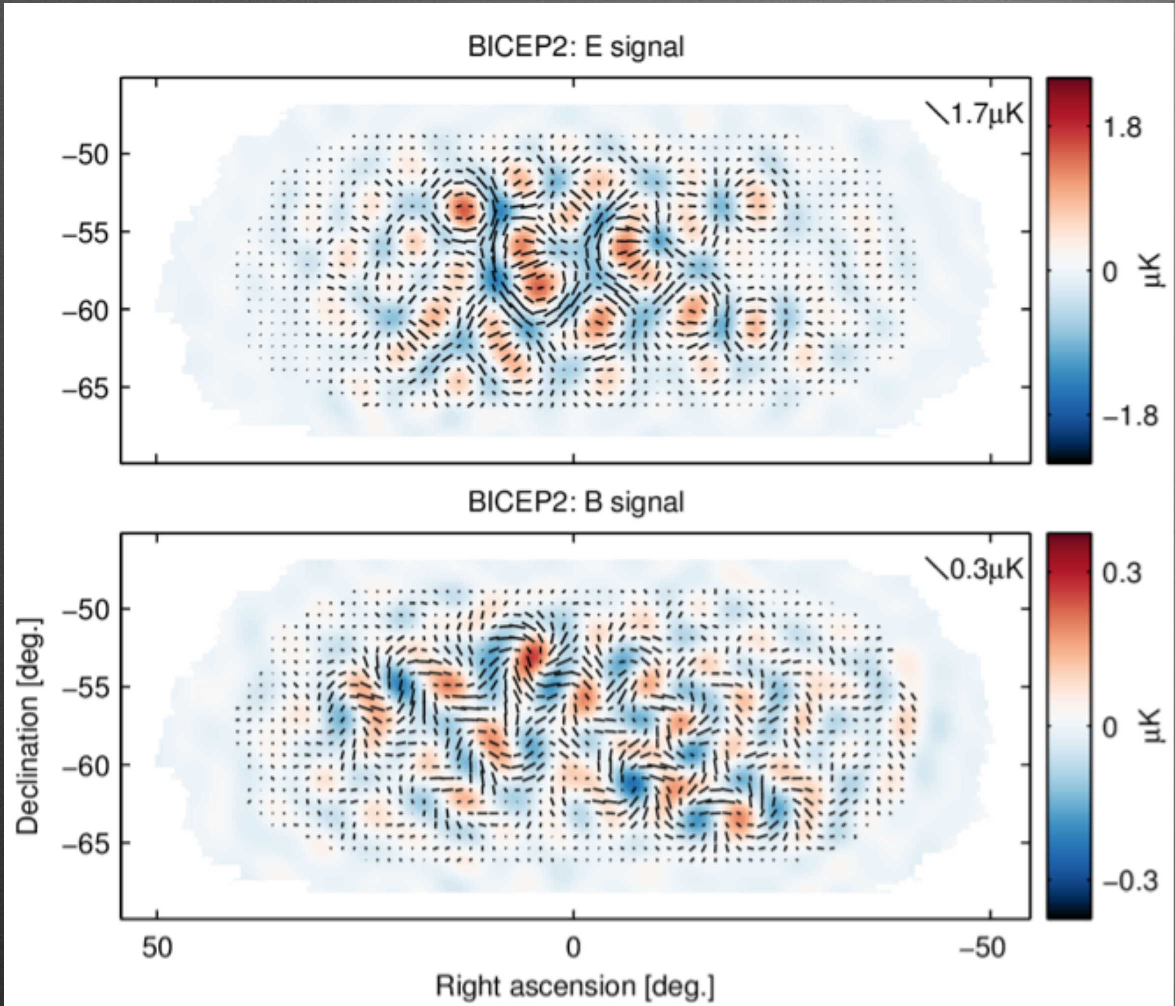
The results from MFs are consistent with those from optimal estimators

B-mode polarization from primordial gravitational wave (GW)

- Polarization pattern separate geometrically into E-mode (divergence-only) and B mode (curl-only)
- Primordial gravitational wave generate B-mode, while density perturbations do not



Hunting primordial B-mode polarization signal



BICEP2 maps at 150GHz

On-going surveys:
POLARBEAR
BICEP/Keck Array,
SPTPol, ACTPol

tensor-to-scalar ratio

$r<0.07$ (95%CL)

Future surveys:
LiteBIRD, PIXIE
CMB-S4

$\sigma(r)\sim0.001$

Origin of primordial GW

- Many studies assume that the detection of primordial B-mode comes from the quantum fluctuations of vacuum during the cosmic inflation

Is the assumption always true ?

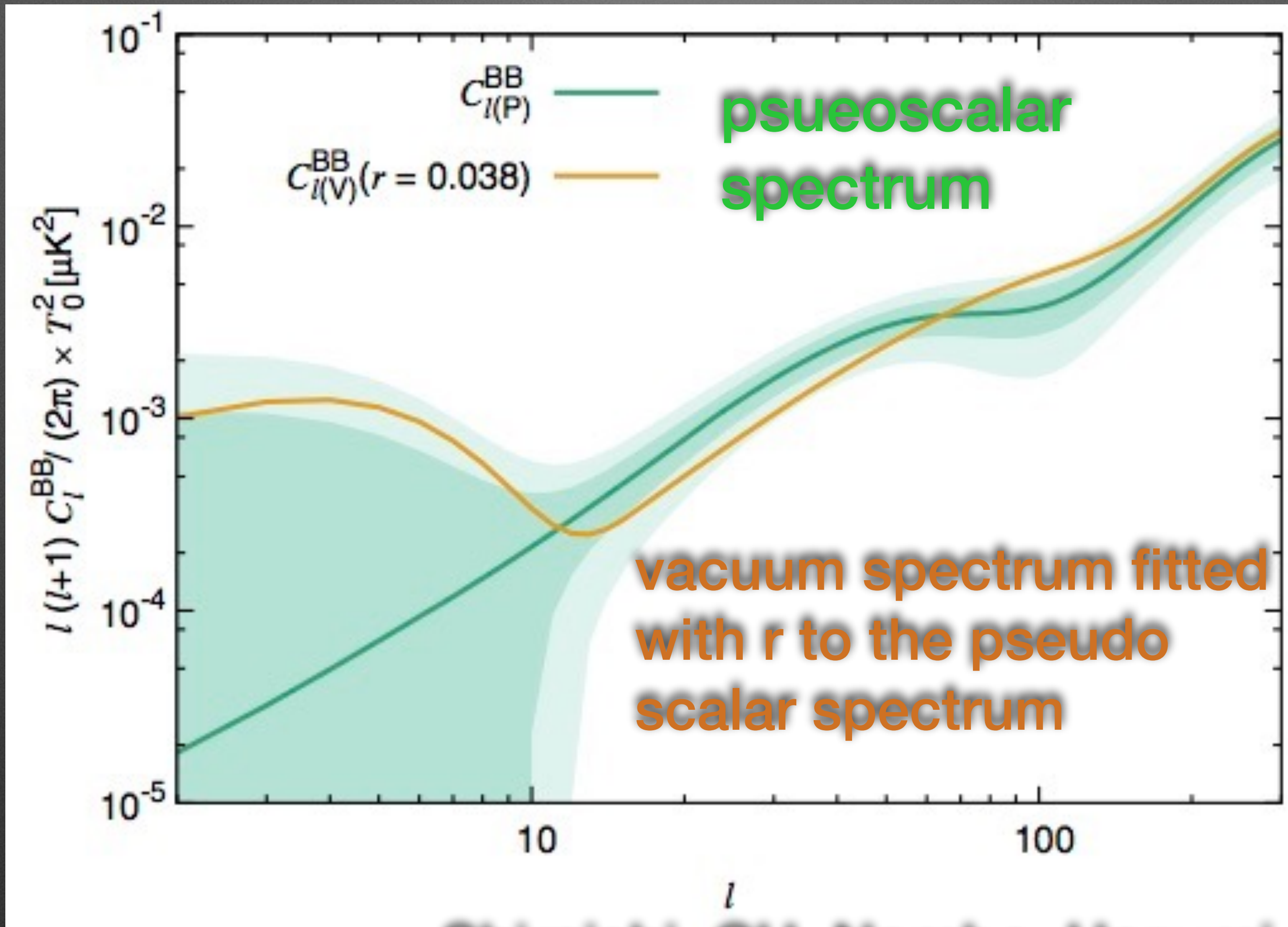
- GW could be sourced by particles produced during inflation

e.g., gauge field sources chiral GWs via a pseudo scalar coupling (Namba et al. 2016)

Validity of the assumption should be tested

B-mode power spectrum

pseudo-scalar model (Namba et al. 2016)

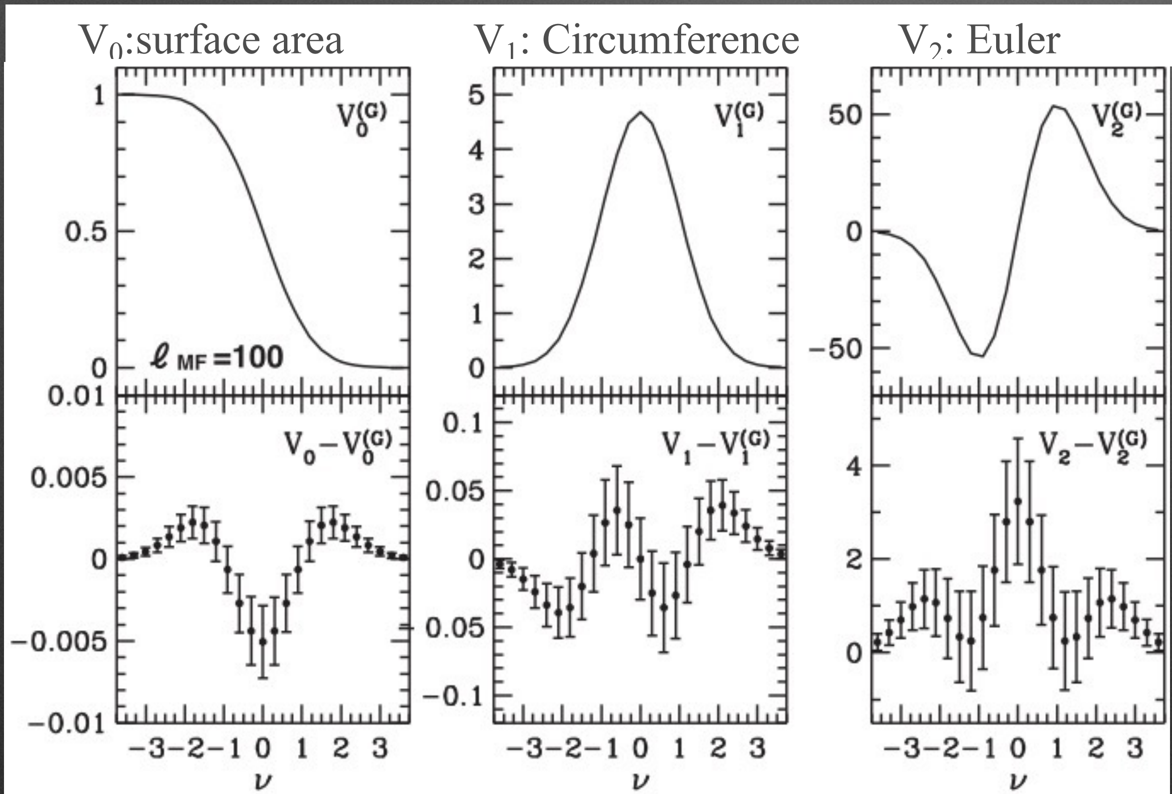


Shiraishi, CH, Namba, Hazumi, Namikawa 2016

Vacuum and pseudoscalar model spectrum are compatible within 1-sigma

Non-Gaussianity in CMB B-mode polarization map

- The vacuum mode is nearly Gaussian, however the mode sourced by other fields could have large NG



The source-field NG in B-mode map is detectable at LiteBIRD experiment.

NG in B-mode map is useful to reveal the origin of the primordial B-mode

Shiraishi, CH, Namba, Hazumi, Namikawa 2016

Tensor Minkowski functionals

- Tensorial generalization of usual (scalar) MFs (Alesker 1999, Beispert et al. 2002)
- Tensor MFs of rank 2 are defined as

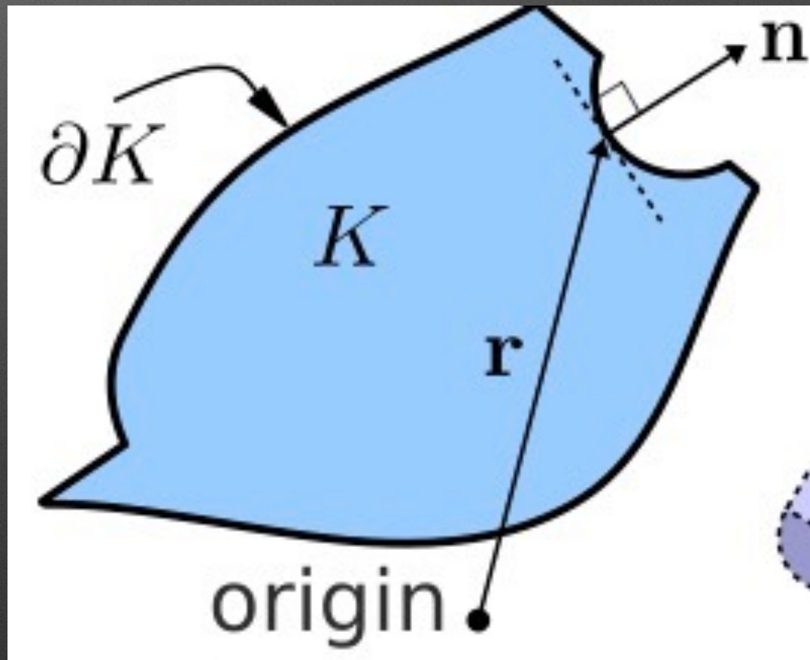
$$W_0^{2,0}(K) = \int_K \mathbf{r} \otimes \mathbf{r} \, dA$$

$$W_1^{a,b}(K) = \frac{1}{4} \int_{\partial K} \mathbf{r}^a \otimes \mathbf{n}^b d\ell$$

$$W_2^{a,b}(K) = \frac{1}{2\pi} \int_{\partial K} \kappa (\mathbf{r}^a \otimes \mathbf{n}^b) d\ell$$

a+b=2

$$(\mathbf{A} \otimes \mathbf{B})_{ij} = (A_i B_j + A_j B_i)/2$$



Schroder-Turk et al. 2010

Anisotropy measures

- If the field is isotropic, the tensor MFs should be isotropic, i.e., the eigenvalues of the tensor are all equal

- Deviations from isotropy can be measured from the ratio of the eigenvalues

$$\beta = |\xi_1/\xi_2| \quad (|\xi_1| \leq |\xi_2|)$$

ξ_i are the eigenvalues of a tensor MF

- tensor MFs quantifies the anisotropy of shape on different scales, which cannot be captured by the usual scalar MFs

Summary

- Minkowski Functionals (MFs) are morphological descriptors satisfying motion-invariant, additive and continuous conditions
- Analytical formulae of MFs are derived in random Gaussian fields and also in weakly non-Gaussian fields
- MFs have been applied to cosmological random fields such as CMB temperature/polarization maps as a probe of NG from morphological point of view
- Tensor MFs become a novel probe of anisotropy and can be used to test the isotropy of our Universe