Basic properties of unit hypercube

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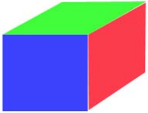
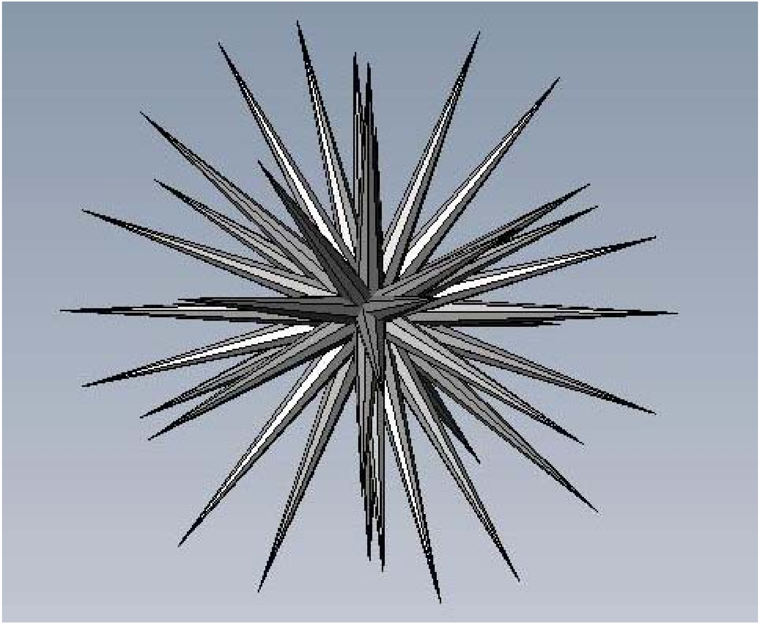
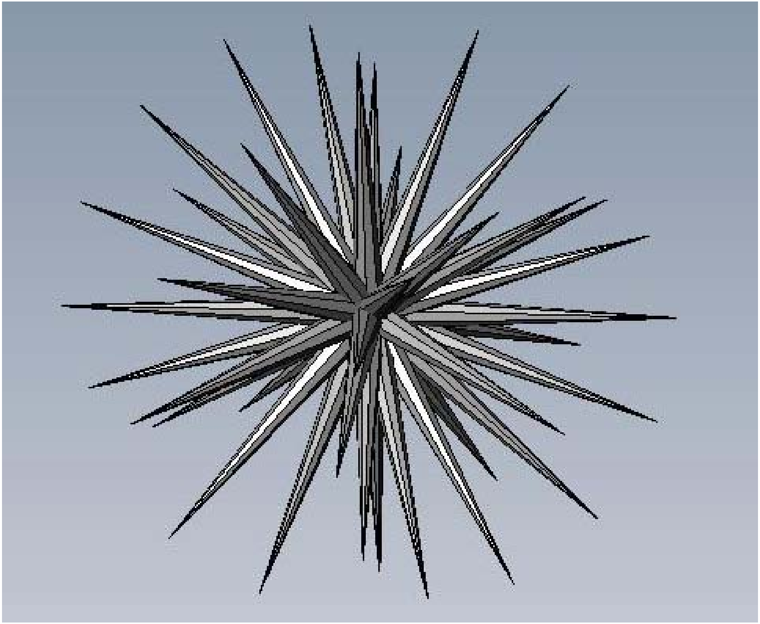
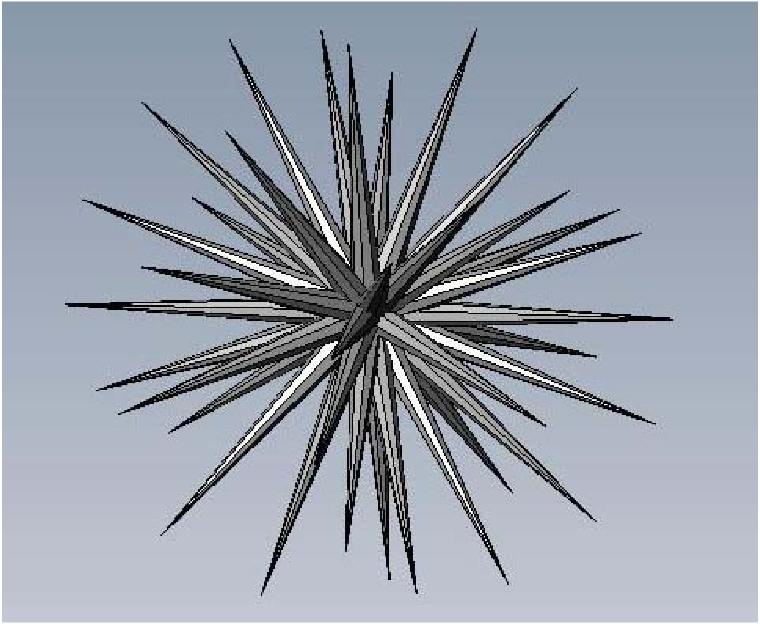
How much are cubes differed at different dimensions?

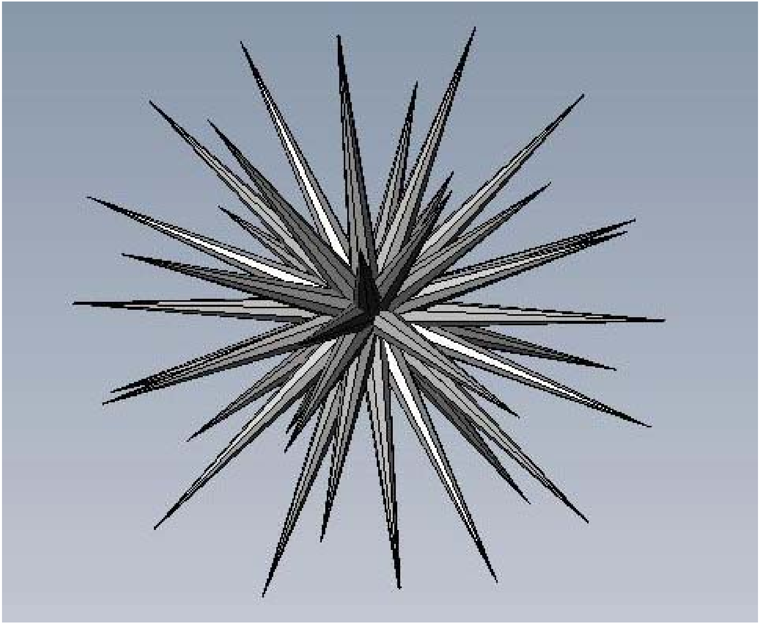


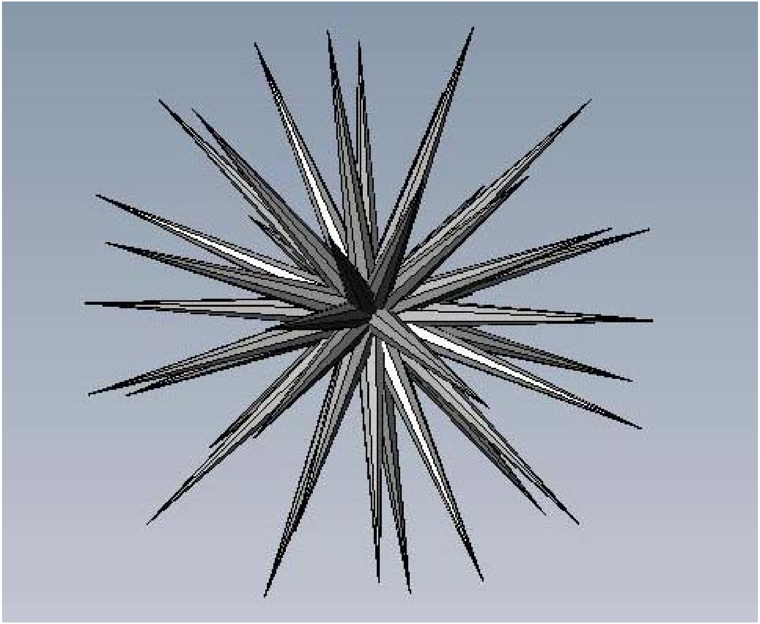
$[0,1]^2$ $[0,1]^3$ $[0,1]^8$





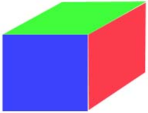




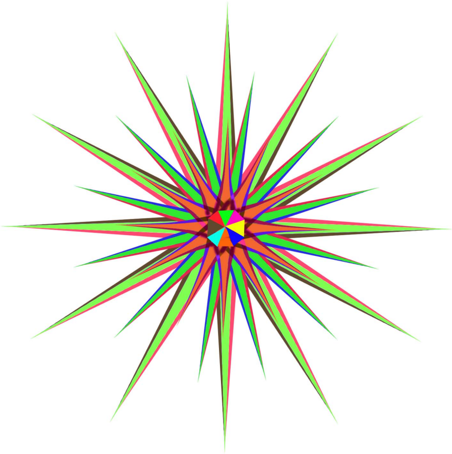


How much are cubes differed at different dimensions?

$[0,1]^2$



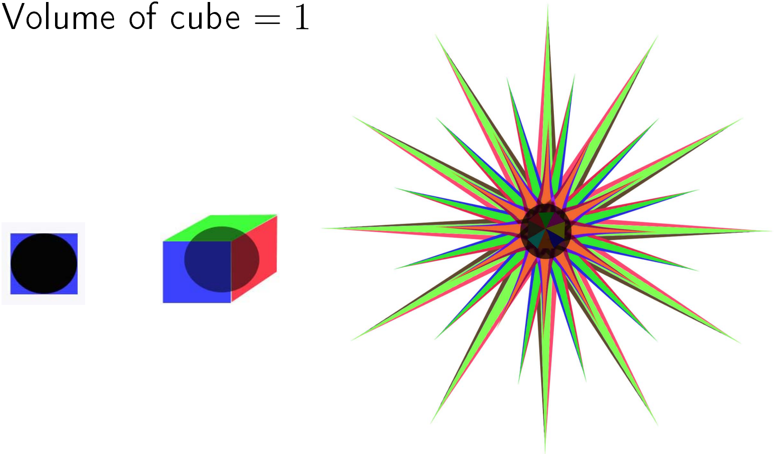
$[0,1]^3$



$[0,1]^8$

Volume of inscribed ball

Volume of cube $= 1$

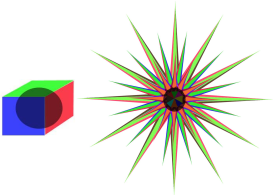


$V_2 \approx 0.78 \quad V_3 \approx 0.52 \quad V_8 = \left.\frac{\pi^{d/2} r^d}{\Gamma(1+d/2)}\right|_{\substack{d=8\\ r=0.5}} \approx 0.016$

Features of cube

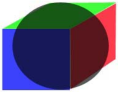
In high-dimensional space

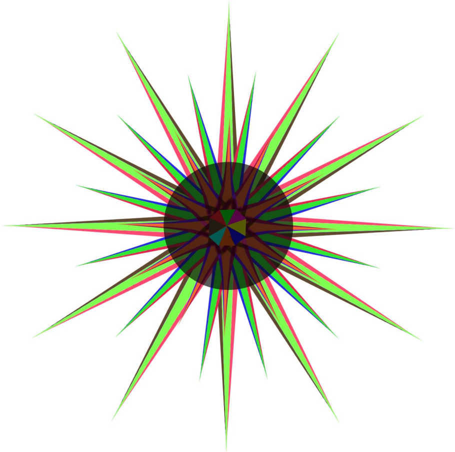
- The 'middle' of cube is empty.
- The cube is a 'union' of its corners.
- The 'average' radius of the cube is about $\sqrt{\frac{d}{2\pi e}}$. Note that the distance from the center to the middle of cube's facets is 0.5 for any dimension d.



Radius of ball of unity volume

Volume of cube $= 1$





$$r_2 \approx 0.56 \quad r_3 \approx 0.62 \quad r_8 \approx \sqrt{\frac{d}{2\pi e}}\Bigg|_{d=8} \approx 0.84$$

Projection of ball of unity volume

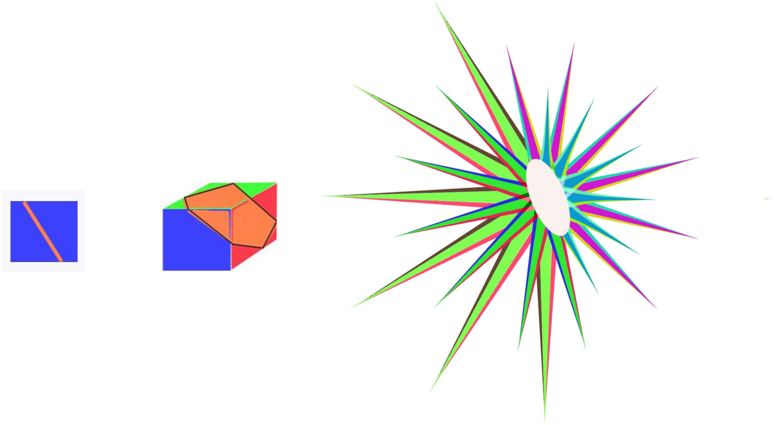
If one project the mass distribution of the ball of volume 1 onto a single direction, one get a distribution that is approximately Gaussian with variance $1/(2\pi e)$.

Note that

- the variance does not depend upon the dimension d,
- the radius of ball of volume 1 grows like $\sqrt{\frac{d}{2\pi e}}$.

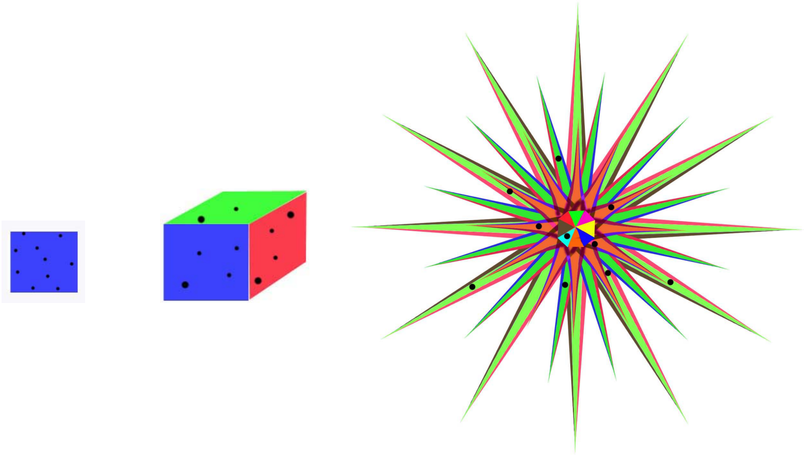
Most of the volume of the ball lies near its surface.

Sections of cube



The cube in $\mathbb{R}^d$ has almost spherical sections whose dimension is roughly $\log d$ and not more.

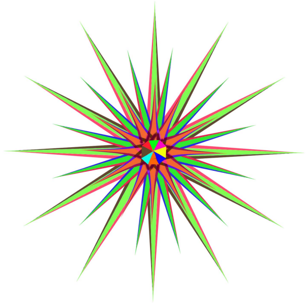
10 uniform points



Distance between points grows very fast

Conclusion

- 2D and 3D intuition might lead us astray in high-dimensional spaces.
- The cube is a bad approximation to the ball (the distance is at most $\sqrt{d}/2$).



Reference

K. Ball (1997) An elementary introduction to modern convex geometry