Linear generalized Kalman–Bucy filter

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1 Introduction

The linear generalized Kalman–Bucy filter problem [1] is studied. A signal and a noise are independent stationary auto-regressive processes with orders exceeding 1. In the frames of the recurrent algorithm, equations for the filter and its error are delivered. A direct algorithm is also proposed. The advantages and the locks of both algorithms are discussed. Numerical examples are given.

2 Statement of the problem

Let an observed process $\zeta_t = \theta_t + \eta_t$ be the sum of two independent processes consisting of a signal θ_t and a noise η_t. We assume that the processes θ_t and η_t are stationary autoregressive sequences of orders n and m, respectively,

$$\sum_{k=0}^{n} a_k\theta_{t-k} = \sigma_1\varepsilon_1(t), \quad a_0 = 1, \qquad \sum_{k=0}^{m} b_k\eta_{t-k} = \sigma_2\varepsilon_2(t), \quad b_0 = 1. \tag{1}$$

The random errors $\varepsilon_i,\ i,j = 1,2$ are such that $\mathbf{E}\varepsilon_i(t) = 0,\ \mathbf{E}\varepsilon_i(t)\varepsilon_j(s) = \delta_{ts}\delta_{ij}$ where δ_{kj} is the Kronecker delta. The moduli of the roots of the characteristic polynomials $a(z) = \sum_{k=0}^{n} a_k z^{n-k}$ and $b(z) = \sum_{k=0}^{m} b_k z^{m-k}$ are less than 1.

The Kalman–Bucy filter problem consists in the prognosis of the process θ_t at $t \geq 0$ by using observations of the process ζ_t at $t \geq 0$. In [2] the case $m = n = 1$ is studied. Here we investigate the more general case $n \geq 1,\ m \geq 1,\ n + m > 2$.

3 Recurrent relations of the Kalman-Bucy filter

We denote the process θ_t, estimate and its error with respect to the σ-algebra F_t^ζ,

$$\mu_t = \mathbf{E}(\theta_t|F_t^\zeta), \quad \gamma_t = \mathbf{E}[(\theta_t - \mu_t)^2|F_t^\zeta], \qquad F_t^\zeta = \sigma\{\omega : \zeta_0, \dots, \zeta_t\}. \tag{2}$$

The recurrent relations are obtained in the following form

$$\mu_{t+1} = \sum_{k=0}^{w-1} A_{t,k}\mu_{t-k} + \sum_{k=0}^{m-1} B_{t,k}\zeta_{t-k}, \quad \gamma_{t+1} = C_t, \tag{3}$$

where $w = \max\{n, m\}$ and $A_{t,k}, B_{t,k}, C_t$ are the non-linear functions of the coefficients $a_j,\ b_j,\ \sigma_j$ in Eqs. (1), and of $\gamma_i,\ t+1-w \leq i \leq t$. Relations (3) are unacceptable at $0 \leq t \leq w-2$, and for such t a direct algorithm shall be used.

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4 The direct algorithm

For $t \geq 0$ the Kalman–Bucy filter μ_t in the linear approximation is

$$\mu_t^d = \mathbf{E}(\theta_t | F_t^\zeta) = \sum_{k=0}^{t} \alpha_k^{(t)} \zeta_{t-k} \tag{4}$$

and we should choose the coefficients $\alpha_k^{(t)}$ so that the error $\gamma_t^d = \mathbf{E}(\theta_t - \mu_t^d)^2$ be minimum. That leads us to the equations

$$\sum_{p=0}^{t} \alpha_p^{(t)} R_\zeta(k-p) = R_\theta(t-k), \quad k = 0, 1, \ldots, t, \qquad R_\zeta(t) = R_\theta(t) + R_\eta(t), \tag{5}$$

and to the minimum value of $\gamma_t^d = R_\theta(0) - \mathbf{E}((\mu_t^d)^2) = R_\theta(0) - \sum_{k=0}^{t} \alpha_k^{(t)} R_\theta(t-k)$. Here $R_\theta(t)$ and $R_\eta(t)$ are to be found from the Yule–Walker equations [3].

5 Discussion

The direct algorithm may be used for all t, and not only for initial values of t.

It is proved that the direct algorithm converges as $t \to \infty$. By examples we show that the recurrent algorithm sometimes diverges.

For $n + m \leq 3$ the results of the recurrent and of the direct algorithm exactly coincide. In the remaining cases (excluding $t < w$) $\gamma_t^r > \gamma_t^d$, where γ_t^r and γ_t^d are the errors of the recurrent and of the direct algorithm, respectively.

The relations $\alpha_k^t \to \alpha_k^\infty < \infty$ for a fixed k, and $\alpha_k^\infty \to 0$ as $k \to \infty$ are valid. Therefore, if we want to use the direct algorithm for large values of t, it is possible to avoid the solution of Eqs. (5) of order t. We choose a small ε (say, $\varepsilon = 10^{-3}$), and find τ such that $|\alpha_k^{(t)}| < \varepsilon$ for all $t > \tau$. Then we may use the approximate equations (4) with summation from 0 to τ and with $\alpha_k^{(t)} = \alpha_k^{(\tau)}$.

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References

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