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A stochastic model for the MHD-Burgers system

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The aim of this article is to construct a stochastic representation of the Cauchy problem solution for a class of systems of nonlinear parabolic equations. In other words we aim to reduce the original Cauchy problem to a certain stochastic problem and moreover to construct the required solution via this stochastic system. The systems under consideration can be treated as systems of conservation laws arising in physics, chemistry, biology and other fields and were studied by many authors (see [1] – [3] and references there). We suggest an alternative interpretation for systems of this class and consider them as systems of nonlinear forward Kolmogorov equations for some nonlinear Markov processes. At the first step we find generators of these Markov processes. Unfortunately it appears that we need not these processes but their time reversal and their multiplicative functionals. We illustrate our approach studying as an example the Cauchy problem of the MHD-Burgers system.

Consider a PDE system which describe hydrodynamics in magnetic field including the MHD equation

$$\frac{\partial u_1}{\partial t} + \frac{\partial (u_1 u_2)}{\partial x} = \frac{\sigma^2}{2}\frac{\partial^2 u_1}{\partial x^2}, \quad u_1(0,x) = u_{10}(x) \tag{1}$$

and the Burgers equation with pressure provided by the magnetic field

$$\frac{\partial u_2}{\partial t} + \frac{1}{2}\frac{\partial (u_1^2 + u_2^2)}{\partial x} = \frac{\mu^2}{2}\frac{\partial^2 u_2}{\partial x^2}, \quad u_2(0,x) = u_{20}(x). \tag{2}$$

We construct a probabilistic representation of a weak solution $u = (u_1, u_2)$ to (1), (2) that is of a solution u which satisfy the integral identities

$$\int_0^T \int_R u_m(\theta,x)\left[\frac{\partial h_m(\theta,x)}{\partial \theta} + (\mathcal{A}_m^u + B_m^u)h_m(\theta,x)\right]dxd\theta +$$

$$+\int_R u_{m0}(x)h_m(0,x)dx = 0, \quad m = 1,2,$$

for any test function $h \in C_0^\infty([0,T)\times R)$. Here $\sigma_1 = \sigma, \sigma_2 = \mu$,

$$\mathcal{A}_m^u h_m = \frac{1}{2}\sigma_m^2\frac{\partial^2 h_m}{\partial x^2}, \quad B_1^u h_1(x) = u_2\frac{\partial h_1}{\partial x}, \quad B_2^u(x)h_2(x) = \left[\frac{u_1^2}{2u_2} + \frac{1}{2}u_2\right]\frac{\partial h_2}{\partial x}.$$

Along with PDE system (1),(2) we consider a stochastic system of the form

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$$d\hat{\xi}^m(\theta) = -\sigma_m dw(\theta), \quad \hat{\xi}^m(0) = x, \tag{3}$$

$$\tilde{\eta}^m(t) = exp\left\{\int_0^t C_m^u(\psi_{\theta,t}(x))dw(\theta) - \frac{1}{2}\int_0^t [C_m^u]^2(\psi_{\theta,t}(x))d\theta\right\} \tag{4}$$

$$u_m(t,x) = E[\tilde{\eta}^m(t)u_{0m}(\hat{\xi}^m(t))], \quad m = 1,2. \tag{5}$$

where $\hat{\xi}_{0,x}(t) = \psi_{0,t}(x)$ and

$$C_1^u(x) = \frac{1}{\sigma_1}u_2(\theta,x), \quad C_2^u(x) = \frac{1}{2\sigma_2}\left[u_2(\theta,x) + \frac{u_1^2(\theta,x)}{u_2(\theta,x)}\right].$$

Connections between (1),(2) and (3)–(5) are described in the following assertions.

Theorem 1.*Assume that there exists a unique weak solution $u = (u_1, u_2)$ to (1), (2) which is strictly positive, bounded and differentiable. Then functions $u_m(t,x)$ admit a representation of the form (5).*

We can prove as well an alternative statement.

Theorem 2.*Assume that there exists a unique solution $\hat{\xi}^m(\theta), \tilde{\eta}(\theta), u_m(t,x)$ to (3)– (5) and u_m are strictly positive, bounded and differentiable. Then functions $u_m(t,x)$ of the form (5) satisfy (1), (2) in a weak sense.*

Finally we can prove one more assertion.

Theorem 2.*Assume that u_{m0}, $m = 1,2$ are strictly positive, bounded and differentiable. Then there exists a solution to the stochastic system (3)– (5), functions $u_m(s,x)$ are strictly positive, bounded and differentiable and thus they satisfy (1), (2) in a weak sense.*

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