Autoregressive Negative Binomial Processes

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Abstract

We start by studying first-order autoregressive negative binomial (NBD) processes. We then compare maximum likelihood and moment based estimators of the parameters of the NBD INAR(1) model and show that the degree of dependence has significant effect on the quality of the estimators. Finally, we construct NBD processes with long-range dependence by using the NBD INAR(1) processes as basic building blocks.

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1 INTRODUCTION AND BACKGROUND

In this section we provide a short review of relevant results. We present the negative binomial distribution and discuss the conditions required to obtain stationary non-negative integer-valued processes and finally consider techniques of parameter estimation . The Poisson and NBD autogressive processes appear as the main examples.

1.1 Integer valued stochastic processes

Non-Gaussian stochastic first-order autoregressive processes or Ornstein-Uhlenbeck type processes in discrete and continuous time have considerable potential as building-blocks for stochastic models of observational time series from a wide range of fields such as financial econometrics or turbulence (see Barndorff-Nielsen [1, 2]; Barndorff-Nielsen et al. [3–6] and references therein). These processes can be defined in a way

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similar to the classical (Gaussian) Ornstein-Uhlenbeck process, but with the noise or the Wiener process replaced by a more general Lévy process.

The existence of stationary non-Gaussian Ornstein-Uhlenbeck type processes depends critically on whether the stationary distribution of the process is self-decomposable and many of the important distributions encountered in practice satisfy this condition, examples include the Gaussian, gamma, normal inverse Gaussian and hyperbolic distributions (see [1, 4, 7]). However it is well known that there are no self-decomposable discrete distributions (see [8]).

Nevertheless, by defining an alternative form of self-decomposability for non-negative integer-valued random variables, it is possible to construct first-order autoregressive integer-valued processes or INAR(1) processes with discrete marginal distributions such as the Poisson, NBD or compound Poisson distributions (see for example [9–14]).

1.2 The negative binomial distribution (NBD)

The NBD distribution is a two parameter distribution usually defined by the mean m and shape parameter γ, with probabilities p_x given by

$$p_x = \frac{\Gamma(\gamma+x)}{x!\Gamma(\gamma)}\left(1+\frac{m}{\gamma}\right)^{-\gamma}\left(\frac{m}{m+\gamma}\right)^{x} \qquad \begin{array}{c} x=0,1,2,\ldots \\ \gamma>0, \quad m>0 \end{array}.$$

As an alternative to the parameter m, the parameter $\beta = \gamma/m$ is sometimes used. Thus, we shall use either the pair of parameters (γ, m) or (γ, β). In the former case, we shall refer to the distribution as $\mathrm{NBD}_{(\gamma,m)}$ and in the latter case as $\mathrm{NBD}^{(\gamma,\beta)}$. The probability generating function (PGF) of the NBD distribution is

$$G_X(s) = \sum_{x=0}^{\infty} p_x s^x = \left(\frac{\gamma/m+1-s}{\gamma/m}\right)^{-\gamma} = \left(\frac{\beta+1-s}{\beta}\right)^{-\gamma}.$$

1.3 Discrete analogues of self-decomposability

The existence of a stationary AR(1) process requires self-decomposability of the marginal distribution, however no non-degenerate discrete distributions satisfies this criteria. In this subsection we use [8, 15] to consider the self-decomposability of non-negative discrete distributions.

Definition 1.1 (α-thinning) *Let X be a non-negative integer-valued random variable and let $\alpha \in [0,1]$, then the α-thinning of X using the Steutel and Van Harn operator is defined by*

$$\alpha \circ X \stackrel{\text{dist.}}{=} Z_1 + \ldots + Z_X, \tag{1.1}$$

where Z_i $(i = 1, \ldots, X)$ are mutually independent Bernoulli random variables with $P(Z_i = 1) = \alpha$ and $P(Z_i = 0) = 1 - \alpha$. Additionally, the random variables Z_i are independent of X. The PGF $G_{\alpha \circ X}(s)$ of the random variable (1.1) is

$$G_{\alpha \circ X}(s) = G_X(1 - \alpha + \alpha s).$$

In fact the thinning operation need not be restricted to random variables with a Bernoulli distribution; non-negative integer-valued processes with generalized thinning have been discussed in [16].

Definition 1.2 (Discrete infinite-divisibility) *Let X be a non-negative integer-valued random variable, then X is said to be infinitely-divisible if, for every non-negative integer n, the random variable X can be represented as a sum of n independent and identically distributed (i.i.d.) non-negative random variables $X_{n,i}$ $(i = 1, \ldots, n)$; that is,*

$$X \stackrel{\text{dist.}}{=} X_{n,1} + \ldots + X_{n,n}, \quad n = 1, 2, \ldots.$$

(Note that the distributions of X and $X_{n,i}$ $(i = 1, \ldots, n)$ do not necessarily have to be of the same family.) Moreover, a random variable X is said to be discrete infinitely-divisible if the random variables $X_{n,i}$ only take non-negative integer values. A necessary condition for such a random variable X to be well defined is $P(X = 0) > 0$ (to exclude the trivial case we assume $P(X = 0) < 1$).

Definition 1.3 (Discrete self-decomposability) *A non-negative integer-valued random variable X is said to be discrete self-decomposable if for every $\alpha \in (0, 1)$ the random variable X can be written as*

$$X \stackrel{\text{dist.}}{=} \alpha \circ X + X_\alpha \tag{1.2}$$

where the random variables $\alpha \circ X$ and X_α are independent. In terms of the probability generating functions equation (1.2) can be rewritten as

$$G_X(s) = G_X(1 - \alpha + \alpha s) G_{X_\alpha}(s),$$

where $G_{X_{\alpha(s)}}$ is the PGF of X_α.

For a fixed $\alpha \in (0, 1)$, the thinning operation $\alpha \circ X$ is integer-valued and (1.2) therefore implies that the component X_α must also be integer-valued. We note that the distribution of X_α uniquely determines the distribution of X. Moreover, a discrete infinitely-divisible distribution is discrete self-decomposable.

Unless otherwise stated we shall only consider non-negative integer-valued first-order autoregressive processes in discrete time given by $\{X_t;\, t \in \mathbb{Z}\}$. We shall always refer to such a process using the shorthand notation of X_t. Additionally, we shall often write equality of random variables by which we mean equality in distribution.

1.4 The INAR(1) process

In this subsection we review the results of Al-Osh and Alzaid [11] and McKenzie [9, 10] who independently introduced the INAR(1) process in an effort to develop a non-negative integer-valued autoregressive process analogous to the continuous AR(1) model. Al-Osh and Alzaid [11] concentrated on estimating the parameters of an INAR(1) process with a Poisson marginal distribution. They considered the use of conditional ML estimators and standard moment based estimators. McKenzie in [9] considered two types of NBD autoregressive processes derived as analogues to the Gamma autoregressive processes. The two processes can be differentiated by the fact that one INAR(1) process has deterministic thinning and the other INAR(1) process has stochastic thinning. We define the two processes below.

1.4.1 The INAR(1) process with deterministic thinning

Definition 1.4 (The INAR(1) process) *Let X_t be a non-negative integer-valued autoregressive process of the first-order and let $\alpha \in (0, 1)$ be a deterministic thinning parameter. Then the INAR(1) process is defined by*

$$X_t = \alpha \circ X_{t-1} + \varepsilon_t, \tag{1.3}$$

where ε_t $(t \in \mathbb{N})$, termed the noise process, is a sequence of uncorrelated random variables. Given that both X_t and ε_t are independent and have finite means and variances, a stationary process of the above form is well-defined if and only if X_t is discrete self-decomposable.

Note that the noise process is not the same as the innovation process as defined in [17] for a standard AR(1) process. Let e_t be the innovation process for the AR(1) model, then $e_t = X_t - \alpha X_{t-1} \neq \varepsilon_t = X_t - \alpha \circ X_{t-1}$. Note also that if $\alpha = 0$ then X_t is a realization of i.i.d. random variables.

For $0 < \alpha < 1$, a stationary process of the form (1.3) only exists if the marginal distribution of the process is discrete self-decomposable whose generating function can be written in the form

$$G_X(s) = G_X(1 - \alpha + \alpha s)\, G_\varepsilon(s). \tag{1.4}$$

Due to the discrete self-decomposability of the process (implying stationarity) the autocorrelation function only depends on the time interval between events, it is in fact identical to the autocorrelation function of an AR(1) process. Let X_t be an INAR(1) process with finite first and second moments then the autocorrelation function at lag u is given by

$$\rho(X_t, X_{t+u}) = \rho(u) = \alpha^{|u|}, \quad u \in \mathbb{Z}.$$

Example 1.1 *Let the process X_t have a Poisson ($Po(\lambda)$) marginal distribution, then X_t is discrete self-decomposable since the generating function can be written in the form of (1.4) and hence a Poisson INAR(1) process is well defined. Indeed the PGF may be written as*

$$\exp\left(\lambda(s-1)\right) = \exp\left(\lambda\alpha(s-1)\right)\exp\left(\lambda(1-\alpha)(s-1)\right).$$

In terms of the distributions, this is

$$\mathrm{Po}(\lambda) = \mathrm{Po}(\lambda\alpha) * \mathrm{Po}(\lambda(1-\alpha)).$$

where $*$ represents the convolution of two distributions.

Example 1.2 *Let the process X_t have a NBD$^{(\gamma,\beta)}$ marginal distribution, then X_t is discrete self-decomposable since the generating function can be written in the form of (1.4) and hence a NBD INAR(1) process is well defined. Indeed, the PGF may be written as*

$$\left(\frac{\beta+1-s}{\beta}\right)^{-\gamma} = \left(\frac{\beta+\alpha(1-s)}{\beta}\right)^{-\gamma}\left(\frac{\beta+1-s}{\beta+\alpha(1-s)}\right)^{-\gamma}. \qquad (1.5)$$

In terms of the distributions,

$$\mathrm{NBD}^{(\gamma,\beta)} = \mathrm{NBD}^{(\gamma,\beta/\alpha)} * \text{Noise Distribution}.$$

McKenzie in [9] first introduced the NBD INAR(1) process but did not identify the distribution of the ε_t. A method for generating a noise process was however presented, since it was shown that the distribution of ε_t can be represented in the form of the compound Poisson process

$$\varepsilon_t = \sum_{i=1}^{N}(\alpha^{U_i}) \circ Y_i, \qquad \alpha \in (0,1). \qquad (1.6)$$

Here N is Poisson with mean $-\gamma \ln \alpha$, the U_i are uniformly distributed on $(0,1)$ and the Y_i are NBD$^{(1,\beta)}$ random variables. The random variables N, U_i and Y_i, $(i = 1 \dots N)$ are mutually independent.

Maximum likelihood (ML) estimation for parameters of the NBD INAR(1) model requires knowledge of the marginal distribution of the noise process. In Section 3 we prove that the marginal distribution of the noise process is the so-called negative-binomial geometric distribution.

1.4.2 The NBD INAR(1) process with stochastic thinning

As an alternative to the NBD INAR(1) process, McKenzie in [9] proposed a process whereby the ε_t also have a NBD distribution. Assume that there exists a non-negative integer-valued autoregressive process X_t with i.i.d. stochastic thinning parameters A_t with cumulative distribution function (c.d.f.) F_A concentrated on the interval $(0,1)$, then the INAR(1) process with stochastic thinning is defined by

$$X_t = A_t \circ X_{t-1} + \varepsilon_t, \tag{1.7}$$

where for fixed t the A_t, X_{t-1}, and ε_t are independent random variables. If X_t is to form a stationary solution to (1.7) then the PGF of the stationary distribution must satisfy

$$G_X(s) = \int_0^1 G_X(1-y+ys)\, dF_A(y)\; G_\varepsilon(s). \tag{1.8}$$

If A_t follows a Beta$(a, \gamma - a)$ distribution, then the process $A_t \circ X_{t-1}$ follows a NBD$^{(a,\beta)}$ distribution. (The Beta distribution Beta(p,q) distribution used here has density function $f(x) = x^{p-1}(1-x)^{q-1}/B(p,q), \quad p > 0, \quad q > 0, \quad 0 < x < 1.$) Additionally, choosing the distribution of the ε_t to follow a NBD$^{(\gamma-a,\beta)}$ distribution ensures that the process X_t remains stationary. In this case (1.8) becomes

$$\left(\frac{\beta+1-s}{\beta}\right)^{-\gamma} = \left(\frac{\beta+1-s}{\beta}\right)^{-a} \left(\frac{\beta+1-s}{\beta}\right)^{-(\gamma-a)}.$$

In terms of the distributions,

$$\text{NBD}^{(\gamma,\beta)} = \text{NBD}^{(a,\beta)} * \text{NBD}^{(\gamma-a,\beta)}.$$

1.5 Estimating parameters of the NBD INAR(1) process

To estimate parameters in the INAR(1) model, it is standard to use the conditional maximum likelihood estimators (a version of the ML) and conditional least squares estimators suggested in [11]. Savani and Zhigljavsky in [18] considered moment based estimators for the NBD INAR(1) process based on the ergodicity of the process. In this section, we compare the ML estimator with the moment based estimators. In the problem of parameter estimation for the NBD, it is customary to estimate the parameters (γ, m) and therefore to use the NBD$_{(\gamma,m)}$ representation.

1.5.1 Estimating the thinning parameter

As the autocorrelation functions of the AR(1) and INAR(1) processes coincide, we may consider standard estimation methods used in AR(1) models for the estimation of the thinning parameter. The simplest method is based on the Yule-Walker estimators which use the structure of the autocorrelation function at different lags. The estimator of the thinning parameter is given by the solution of the equation $\hat{\rho}(u) = \hat{\alpha}^{|u|}$, $u \in \mathbb{Z}$. Here $\hat{\rho}(u)$ represents the sample autocorrelation function at lag u. Note that a simple estimator for the thinning parameter is $\hat{\alpha} = \hat{\rho}(1)$.

1.5.2 Maximum likelihood estimation

The likelihood function is straightforward to write using the fact that the INAR(1) process is a Markov process. Let $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ be an observed sample from an INAR(1) process and let Θ be the set of parameters for the INAR(1) process then the likelihood function is

$$\begin{aligned} L(\mathbf{x};\ \Theta) &= \mathbb{P}(X_1 = x_1) \prod_{t=2}^{N} \mathbb{P}(X_t = x_t | X_{t-1} = x_{t-1}) \\ &= \mathbb{P}(X_1 = x_1) \prod_{t=2}^{N} \mathbb{P}\left(X_t = \alpha \circ x_{t-1} + \varepsilon_t\right) \\ &= \mathbb{P}(X_1 = x_1) \prod_{t=2}^{N} \sum_{r=0}^{\min(x_t, x_{t-1})} \binom{x_{t-1}}{r} \alpha^r (1-\alpha)^{x_{t-1}-r} \mathbb{P}(\varepsilon_t = x_t - r). \end{aligned} \tag{1.9}$$

Using this expression for the likelihood function, the ML estimator can be computed numerically.

1.5.3 Moment based estimators

Moment based estimators for the INAR(1) process are computed using the fact that the INAR(1) process is an ergodic process which makes it possible to equate moments of a single observed realization to the moments of the stationary distribution.

In the estimation of γ and m for the NBD INAR(1) model, we shall consider the two most widely used methods in practice (see e.g. [19]), specifically the method of moments (MOM) and the zero term method (ZTM), both of which are moment based. Let $\{x_t;\ t = 1, \ldots, N\}$ be a realization of size N from an INAR(1) process then the MOM and ZTM estimators are obtained by equating the sample moments $(\overline{x}, \overline{x^2})$ and $(\overline{x}, \hat{p_0})$ to their theoretical expectations respectively. Here $\overline{x} = \frac{1}{N}\sum_{t=1}^{N} x_t$, $\overline{x^2} = \frac{1}{N}\sum_{t=1}^{N} x_t^2$ and $\hat{p_0} = n_0/N$ where n_0 denotes the number of zeros in a sample. The expected values of the sample moments for a $\mathrm{NBD}_{(\gamma,m)}$ process are

$$E[\overline{x}] = m, \quad E[\overline{x^2}] = \frac{m(\gamma m + \gamma + m)}{\gamma} \quad \text{and} \quad E[\hat{p_0}] = \left(\frac{\gamma}{m+\gamma}\right)^{\gamma}. \tag{1.10}$$

The estimator of the parameter m for both MOM and ZTM is $\hat{m} = \overline{x}$. The MOM estimator of the parameter γ is $\hat{\gamma}_{(MOM)} = \overline{x}^2/(\overline{x^2} - \overline{x}^2 - \overline{x})$, and the ZTM estimator $\hat{\gamma}_{(ZTM)}$ of γ is the solution of the equation $\hat{p_0} = (1 + \overline{x}/z)^{-z}$ in z. Note that there is no closed form solution for $\hat{\gamma}_{(ZTM)}$.

1.6 Structure of the paper

In section 2 we present a new type of NBD process as a generalization of the processes defined by McKenzie [9]. In section 3 we present the marginal distribution of the noise process for the NBD INAR(1) process with deterministic thinning. In section 4 we consider the problem of estimating the shape and scale parameters in a NBD INAR(1) process. Finally in section 5 we construct long-range dependent processes by the aggregation of independent INAR(1) processes using the approach of Barndorff-Nielsen et al. [1, 2, 4], which has been applied in the case of AR(1) models.

2 THE INAR PROCESS WITH MIXED THINNING

In this section we introduce a more general INAR process as a mixture of the two processes described in Section 1.4. We first define a general INAR process with mixed thinning.

Definition 2.1 *Let X_t be a stationary non-negative integer-valued autoregressive process of the first-order with noise process ε_t. Additionally, let the random variables X_t and ε_t be independent and let ε_t form a sequence of uncorrelated random variables. Assume that both processes have finite means and variances. Let $\alpha \in (0,1)$ be a deterministic thinning parameter and A_t be i.i.d. stochastic thinning parameters with c.d.f. F_A concentrated on the interval $(0,1)$. Then the INAR(1) process with mixed thinning is defined by*

$$X_t = \alpha A_t \circ X_{t-1} + \varepsilon_t. \tag{2.1}$$

The generating function of X_t, defined by (2.1), is

$$G_X(s) = \int_0^1 G_X(1 - y\alpha + y\alpha s)\, dF_A(y) \;\; G_\varepsilon(s). \tag{2.2}$$

The structure of the noise process in the case of NBD INAR process with mixed thinning will be studied in Section 3.3. Below we study the correlation function of the general INAR process with mixed thinning.

Proposition 2.1 *Let X_t be an INAR(1) process with mixed thinning (see (2.1)), with thinning parameters α and A_t, where A_t has distribution function F_A with mean $E[A]$. Assume that the process has finite first and second moments, then the autocorrelation at lag u is*

$$\rho(X_t, X_{t+u}) = \rho(u) = (\alpha E[A])^{|u|}, \quad u \in \mathbb{Z}. \tag{2.3}$$

PROOF. Let A_1 and A_2 be two random variables with c.d.f. concentrated on $(0, 1)$, then it is straightforward to show that for any non-negative integer X, the thinning operation satisfies $A_1 \circ A_2 \circ X = A_1 A_2 \circ X$. The process X_t in (2.1) may be written in terms of X_{t-u} as

$$X_t = \left(\prod_{i=0}^{u-1} \alpha A_{t-i} \right) \circ X_{t-u} + \sum_{j=1}^{u-1} \left(\prod_{i=0}^{j-1} \alpha A_{t-i} \right) \circ \varepsilon_{t-j} + \varepsilon_t.$$

The autocovariance at lag u is

$$\begin{aligned} R(u) &= Cov[X_t, X_{t-u}] \\ &= Cov\left[\left(\prod_{i=0}^{u-1} \alpha A_{t-i} \right) \circ X_{t-u} + \sum_{j=1}^{u-1} \left(\prod_{i=0}^{j-1} \alpha A_{t-i} \right) \circ \varepsilon_{t-j} + \varepsilon_t, X_{t-u} \right] \\ &= Cov\left[\left(\prod_{i=0}^{u-1} \alpha A_{t-i} \right) \circ X_{t-u}, X_{t-u} \right] + Cov\left[\sum_{j=1}^{u-1} \left(\prod_{i=0}^{j-1} \alpha A_{t-i} \right) \circ \varepsilon_{t-j} + \varepsilon_t, X_{t-u} \right] \\ &= E\left[\prod_{i=0}^{u-1} \alpha A_{t-i} \right] Var\left[X_{t-u}\right] + 0 = (\alpha E[A])^u \, Var\left[X_{t-u}\right], \quad u \in \mathbb{Z}^+. \end{aligned}$$

Here we have used the fact that for any $t > s$ the pairs (ε_t, X_s) are uncorrelated. Additionally, from the stationarity of the process, we have $Var[X_{t-u}] = Var[X_t]$ and the expression (2.3) for the autocorrelation function of the process then follows.

■

Note that by taking $\alpha = 1$ we obtain the autocorrelation function of (1.7), the INAR process with stochastic thinning; this correlation function is $E[A]^{|u|}, \; u \in \mathbb{Z}$.

3 NOISE PROCESSES IN THE NBD INAR MODELS

In this section we study the distribution of the noise process in the NBD INAR models. We need these distributions, for example, if we decide to use the likelihood function either to estimate parameters or to test hypotheses.

3.1 The negative-binomial geometric distribution

The negative-binomial geometric distribution (NBGD) with parameters (m, p, θ) has PGF

$$G(s) = \theta^m \left[1 - (1-\theta)p(1-qs)^{-1}\right]^{-m}, \tag{3.1}$$

see [20, pp. 459–460]. Here $0 < p \leqslant 1$, $q = 1 - p$, $0 < \theta \leqslant 1$ and $m \geqslant 0$. To generate random variables from the NBGD(m, p, θ) distribution we may use the fact that NBGD(m, p, θ) can be represented as $\mathrm{NBD}^{(\xi, p/(1-p))}$, where ξ is a random variable with distribution $\mathrm{NBD}^{(m, \theta/(1-\theta))}$ (see [20, p. 460]).

The NBGD is a compound Poisson distribution (see (1.6)). Indeed, the PGF (3.1) can be written in the form $G(s) = e^{\lambda\,(G_\phi(s)-1)}$, where

$$\lambda = -m \ln\theta \quad \text{and} \quad G_\phi(s) = \frac{1}{\ln\theta} \ln\left[1 - (1-\theta)p\,(1-qs)^{-1}\right].$$

Here $G_\phi(s)$ is the generating function of the so-called logarithmic-geometric distribution (with parameters q and $1-\theta$), see [20]. This confirms the result stated in [9].

3.2 Noise process in the NBD INAR(1) model with deterministic thinning

The NBD INAR(1) model with deterministic thinning has been discussed in several papers. However, the distribution of the noise ε_t has not been identified althugh methods for simulating such random variables do exist (see equation (1.6)).

Proposition 3.1 *Let X_t be an INAR(1) process with thinning parameter α and let $X_t \sim \mathrm{NBD}^{(\gamma,\beta)}$, then ε_t has the distribution NBGD $(\gamma,\ \beta/(\beta+\alpha),\ \alpha)$ with p.m.f.*

$$P\{\varepsilon_t = x\} = \sum_{k=0}^{\infty} \binom{k+x-1}{x} \left(\frac{\beta}{\beta+\alpha}\right)^k \left(\frac{\alpha}{\beta+\alpha}\right)^x \binom{\gamma+k-1}{k} \alpha^\gamma (1-\alpha)^k, \tag{3.2}$$

where $x = 0, 1, 2 \ldots,\ \alpha \in (0,1), \gamma > 0$ and $\beta > 0$.

PROOF. Note that the generating function of the errors can be written as

$$G_\varepsilon(s) = \left[\frac{\beta+(1-s)}{\beta+\alpha(1-s)}\right]^{-\gamma} = \alpha^\gamma \left[\frac{\beta+(1-s)}{\beta/\alpha+(1-s)}\right]^{-\gamma} = \alpha^\gamma \left[1 - \frac{\beta(1-\alpha)}{(\beta+\alpha)\left(1-\frac{\alpha}{\beta+\alpha}s\right)}\right]^{-\gamma}$$

$$= \alpha^\gamma \left[1-(1-\alpha)\left(\frac{\beta}{\beta+\alpha}\right)\left(1-\frac{\alpha}{\beta+\alpha}s\right)^{-1}\right]^{-\gamma}.$$

This is exactly the form of the generating function of the NBGD(m, p, θ) distribution with $m = \gamma, p = \beta/(\beta+\alpha)$ and $\theta = \alpha$, see (3.1). ■

3.3 Noise process in the NBD INAR(1) model with stochastic thinning

Proposition 3.2 *Let the process X_t have a* NBD$^{(\gamma,\beta)}$ *marginal distribution, then X_t may be represented as a process with mixed thinning so that*

$$X_t = \alpha A_t \circ X_{t-1} + \varepsilon_t,$$

with $A_t \sim$ Beta$(a, \gamma - a)$ *and $\varepsilon_t \sim$* NBD$^{(\gamma-a,\beta/\alpha)} *$ NBGD$(\gamma, \beta/(\beta+\alpha), \alpha)$.

PROOF. Let X_t be the INAR(1) process with mixed thinning. Then the PGF of X_t must satisfy equation (2.2). We have $X_t \sim$ NBD$^{(\gamma,\beta)}$ and $G_X(s) = ((\beta+1-s)/\beta)^{-\gamma}$. In order to satisfy equation (2.2), we require

$$G_X(s) = \int_0^1 G_X(1 - y\alpha + y\alpha s) dF_A(y) G_\varepsilon(s)\,.$$

If $F_A(y)$ is the c.d.f. of the random variables $A_t \sim$ Beta$(a, \gamma - a)$ and $G_\varepsilon(s)$ is the PGF of the random variables $\varepsilon_t \sim$ NBD$^{(\gamma-a,\beta/\alpha)} *$ NBGD$(\gamma, \beta/(\beta+\alpha), \alpha)$, then simple calculus gives

$$\int_0^1 G_X(1 - y\alpha + y\alpha s) dF_A(y) = \left(\frac{\beta+\alpha(1-s)}{\beta}\right)^{-a}$$

$$\text{and} \quad G_\varepsilon(s) = \left(\frac{\beta+\alpha(1-s)}{\beta}\right)^{a-\gamma} \left(\frac{\beta+1-s}{\beta+\alpha(1-s)}\right)^{-\gamma},$$

which implies $\int_0^1 G_X(1 - y\alpha + y\alpha s) dF_A(y) G_\varepsilon(s) = \left(\frac{\beta+1-s}{\beta}\right)^{-\gamma}$ as required. ■

4 ESTIMATION IN THE NBD INAR(1) MODEL

In this section we investigate estimators for the NBD parameters of the NBD INAR(1) model with deterministic thinning. We compare standard moment based estimation methods with the ML estimation method; the moment based estimators considered are the MOM and ZTM estimators. We also derive the asymptotic distributions of the moment based estimators.

4.1 Comparing the estimators

The ML and moment based estimators were defined in Section 1.5. The identification of the distribution of the errors, see (3.2), makes it possible to compute the ML estimators. The likelihood function for a sample $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ from the NBD INAR(1) process is given by equation (1.9) where ε_t are i.i.d. and follow the NBGD(γ, $\beta/(\beta+\alpha)$, α) distribution. The likelihood function for a sample from the NBD INAR(1) process is certainly not trivial; finding explicit solutions for the ML estimators is therefore difficult. It is possible, however, to maximize the likelihood function using global optimization algorithms. Maximum likelihood estimators can therefore be obtained by using numerical techniques.

The table below shows the results of a simulation study comprising $R = 1000$ runs of estimating parameters from NBD INAR(1) samples of size $N = 10000$ and thinning parameter $\alpha = 0.5$. The table shows the empirical coefficient of variation $\widehat{\kappa_\gamma} = \sqrt{N}\sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\gamma}_i - \gamma)^2}/\gamma$ for the $\hat{\gamma}_{(ML)}$, $\hat{\gamma}_{(MOM)}$ and $\hat{\gamma}_{(ZTM)}$ estimators of γ and the empirical coefficient of variation $\widehat{\kappa}_m = \sqrt{N}\sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{m}_i - m)^2}/m$ for $\hat{m}_{ML}$ (ML) and $\hat{m} = \bar{x}$ (moment based) estimators of m.

In [18] we have investigated, in the case of an i.i.d. NBD sample, the efficiency of the MOM and ZTM estimators relative to the ML estimators, as a function of the parameter values. Note that for an i.i.d. NBD sample, the MOM, ZTM and ML estimators for m are identical. Unexpectedly, in the case of INAR(1) model, the ML estimator obtained by numerically maximizing the expression (1.9) and the sample mean provide different estimators for m. The ML estimator has a consistently lower empirical coefficient of variation in comparison to the moment based estimators.

γ	m	$\kappa_{\hat{\gamma}_{(ML)}}$	$\kappa_{\hat{\gamma}_{(MOM)}}$	$\kappa_{\hat{\gamma}_{(ZTM)}}$	$\kappa_{\hat{m}_{ML}}$	$\kappa_{\hat{m}}$
0.5	0.5	4.30	9.04	6.37	3.29	3.49
	1	3.14	5.63	4.32	2.79	2.97
	5	2.27	3.51	3.16	2.58	2.61
1	0.5	11.38	15.57	12.74	2.61	2.84
	1	3.86	8.28	5.18	2.37	2.51
	5	2.16	3.57	3.26	1.93	1.96
3	0.5	26.84	32.59	34.07	2.67	2.90
	1	18.45	21.86	20.85	1.83	1.97
	5	2.80	5.15	4.83	1.20	1.20

Table 4.1: Empirical coefficient of variation for estimators for m and k in a simulation study comprising $R = 1000$ runs of estimating parameters from NBD INAR(1) samples of size $N = 10000$ and thinning parameter $\alpha = 0.5$

4.2 Asymptotic normality of sample moments

In this section, we present the asymptotic variances and covariances of the sample statistics used in MOM and ZTM estimators. The proofs follow by standard calculations of the first and second moments of the sample, see, for example, [18]. Note that the asymptotic normality of the sample statistics follows from the standard univariate central limit theorem for Markov chains, see e.g. [21], and the fact that two-variate normality of a random vector is equivalent to univariate normality for any linear combination of the two variables. Below, let X be a random variable with the same distribution as X_t $(t = 0, 1, \ldots)$.

Proposition 4.1 *Let $\{x_t;\ t = 1, 2, \ldots, N\}$ be a sample realization from an INAR(1) process with thinning parameter α then $\overline{x}$ and $\overline{x^2}$ have a joint asymptotic normal distribution given by*

$$\sqrt{N}\begin{pmatrix} \overline{x}-E[X] \\ \overline{x^2}-E[X^2] \end{pmatrix} \sim \mathcal{N}\left[0, \Sigma_{(\overline{x},\overline{x^2})}\right].$$

Here $\Sigma_{(\overline{x},\overline{x^2})}$ is the covariance matrix of $\overline{x}, \overline{x^2}$ with entries

$$\begin{aligned} Var[\overline{x}] &= \left(\frac{1+\alpha}{1-\alpha}\right) Var[X], \\ Var[\overline{x^2}] &= \left(\frac{1+\alpha^2}{1-\alpha^2}\right) Var[X^2] + \frac{2\alpha}{1-\alpha^2}(1+2E[X])Cov[X,X^2], \\ Cov[\overline{x},\overline{x^2}] &= \frac{1+\alpha+\alpha^2}{1-\alpha^2}\ Cov[X,X^2] + \frac{\alpha}{1-\alpha^2}(1+2E[X])Var[X]. \end{aligned}$$

Proposition 4.2 *Let $\{x_t;\ t = 1, 2, \ldots, N\}$ be a sample realization from an INAR(1) process with thinning parameter α, then $\overline{x}$ and $\hat{p_0}$ have a joint asymptotic normal distribution given by*

$$\sqrt{N}\begin{pmatrix} \overline{x}-E[X] \\ \hat{p_0}-p_0 \end{pmatrix} \sim \mathcal{N}\left[0, \Sigma_{(\overline{x},\hat{p_0})}\right].$$

Here $\Sigma_{(\overline{x},\hat{p_0})}$ is the covariance matrix of $\overline{x}, \hat{p_0}$ with entries

$$Var[\overline{x}] = \left(\frac{1+\alpha}{1-\alpha}\right) Var[X],$$

$$Var[\hat{p_0}] = p_0(1-p_0) + 2\lim_{N\to\infty}\left\{\sum_{r=1}^{N-1}\left(1-\frac{r}{N}\right)\left[p_0\prod_{j=0}^{r-1}G_\varepsilon\left(1-\alpha^j\right) - p_0^2\right]\right\},$$

$$Cov[\overline{x},\hat{p_0}] = -p_0E[X] + \lim_{N\to\infty}\sum_{r=1}^{N-1}\left\{\left(1-\frac{r}{N}\right)\left[(1-\alpha^r)\,G'_X\,(1-\alpha^r)\prod_{j=0}^{r-1}G_\varepsilon\left(1-\alpha^j\right) - (1+\alpha^r)\,p_0E[X]\right]\right\}$$

and $G_X(s)$ and $G_\varepsilon(s)$ represent the generating functions of X and ε respectively.

4.3 Asymptotic normality of estimators

To obtain the asymptotic distribution of the estimators, we may use the results of [22] establishing the asymptotic normality of functions of asymptotically normal vectors. Specifically, any vector $f(X_n) = (f_1(X_n), \ldots, f_m(X_n))$, which is a function of a vector of statistics $X_n = (X_1^{(n)}, \ldots, X_k^{(n)}) \sim \mathcal{N}(\mu, b_n^2 \Sigma)$ with $b_n \to 0$ as $n \to \infty$ and covariance matrix Σ, is also asymptotically normally distributed, i.e.

$$f(X_n) \sim \mathcal{N}(f(\mu), b_n^2 D \Sigma D'). \tag{4.1}$$

Here $D = [\partial f_i / \partial X_j]_{i=1,\ldots m, j=1,\ldots,k}$ is the matrix of partial derivatives evaluated at μ. We now present the asymptotic distributions for the MOM and ZTM.

Proposition 4.3 *Let $\{X_t;\ t = 1, \ldots, N\}$ be a NBD$_{(\gamma,m)}$ INAR(1) process with thinning parameter α, then the MOM estimators $(\hat{\gamma}, \hat{m})$ have a joint asymptotical normal distribution with mean (γ, m) and covariance matrix*

$$\begin{bmatrix} Var[\hat{\gamma}] & Cov[\hat{\gamma}, \hat{m}] \\ Cov[\hat{\gamma}, \hat{m}] & Var[\hat{m}] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \left(\frac{1+\alpha^2}{1-\alpha^2}\right) \frac{2\gamma(\gamma+1)(m+\gamma)^2}{m^2} & \frac{2\alpha(m+\gamma)}{1-\alpha} \\ \frac{2\alpha(m+\gamma)}{1-\alpha} & \left(\frac{1+\alpha}{1-\alpha}\right) \frac{m(m+\gamma)}{\gamma} \end{bmatrix}. \tag{4.2}$$

Unlike the method of moments estimators there is no simple asymptotic expression for the covariance matrix for $(\hat{\gamma}, \hat{p_0})$ using the zero term method. Analytic expressions may however be obtained by using equation (4.1) and the following proposition.

Proposition 4.4 *Let $\{X_t;\ t = 1, \ldots, N\}$ be a NBD$^{(\gamma,m)}$ INAR(1) process with thinning parameter α, then the ZTM estimators have a joint asymptotical normal distribution with mean (γ, m) and covariance matrix given by $b_N^2 D \Sigma D'$, where $b_N^2 = 1/N$ and Σ is the covariance matrix of $(\bar{x}, \hat{p_0})$ (see Proposition 4.2) and*

$$D = \begin{bmatrix} \frac{\gamma}{m+(\gamma+m)\log\left(\frac{\gamma}{\gamma+m}\right)} & \frac{(\gamma+m)^{\gamma+1}\gamma^{-\gamma}}{m+(\gamma+m)\log\left(\frac{\gamma}{\gamma+m}\right)} \\ 1 & 0 \end{bmatrix}. \tag{4.3}$$

Example 4.1 *Figures 4.1 and 4.2 show the theoretical (asymptotic normalised) confidence ellipses produced by the MOM and ZTM estimators for $(\hat{\gamma}, \hat{m})$ at a fixed significance level for different values of the thinning parameter α. Here $\hat{\gamma}$ and $\hat{m}$ are represented on the horizontal axis and vertical axis respectively. Starting from the ellipse in the center outwards, the ellipses correspond to estimators when $\alpha = 0,\ 0.4$ and 0.8.*

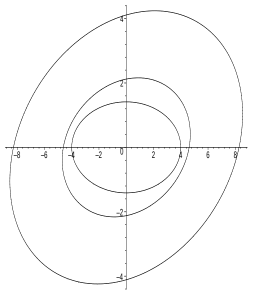

Figure 4.1: Method Of Moments

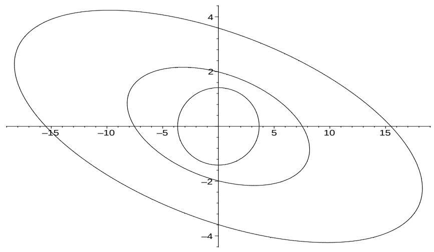

Figure 4.2: Zero Term Method

5 LONG-MEMORY INTEGER VALUED PROCESSES

A process is often said to be long-range dependent or have long-memory if the process has non-summable correlations or if the spectral density has a pole at the origin. There are various statistical definitions of long-memory and they are not all equivalent. A thorough review on long-range dependence has been made in [23, 24].

Barndorff-Nielsen et al. in [2–5] and Leonenko in [25] constructed a stationary long-memory normal-inverse Gaussian (NIG) process in continuous time by the superposition (or aggregation) of short-memory Ornstein-Uhlenbeck type processes with NIG marginal distributions. For suitable parameters of the individual short-memory NIG processes, each with the same autocovariance function, the aggregated process was shown to have long-memory with autocovariance function of the form

$$R(u) \simeq L(u)u^{-2(1-H)}, \quad H \in (1/2, 1)\,, u \in \mathbb{R} \quad \text{as} \quad u \to \infty,$$

where H is the long-memory (or Hurst) parameter and $L(u)$ is a slowly varying function.

5.1 Long-memory integer valued processes

In this section we construct a long-memory non-negative integer-valued process using the approach in [1] by the aggregation ($X_t = \sum_{k=1}^{\infty} X_t^{(k)}$) of a sequence of stationary and independent INAR(1) processes $X_t^{(k)}$ ($k = 0, 1, 2, \ldots$) of the form

$$X_t^{(k)} = \alpha_k \circ X_{t-1}^{(k)} + \varepsilon_t^{(k)}, \quad k = 1, 2, ... \quad t \in \mathbb{Z}. \tag{5.1}$$

Moreover we present conditions required in order to construct long-memory processes with Poisson and NBD marginal distributions and show some simulation results of the autocovariance function and spectral density.

Proposition 5.1 *Let $X_t^{(k)}$ $(k = 1, 2, \ldots)$ be independent INAR(1) processes with mean $\mu_{X_k} < \infty$, variance $\sigma^2_{X_k} < \infty$ and thinning parameter α_k. Let $X_t = \sum_{k=1}^{\infty} X_t^{(k)}$ be the aggregation of the INAR(1) processes such that $E[X_t] = \sum_{k=1}^{\infty} \mu_{X_k} < \infty$. If $\sigma^2_{X_k}$ and α_k are of the form*

$$\sigma^2_{X_k} = \frac{c_1}{k^{1+2(1-H)}}, \quad \alpha_k = \exp\{-c_2 \,/\, k\} \tag{5.2}$$

with some positive constants c_1, c_2 and $1/2 < H < 1$, then the limiting aggregated processes X_t is a well defined process in the L^2 sense with long-memory (or Hurst) parameter H. The autocovariance function and the spectral density of the process are given by equations (5.4) and (5.5) below.

PROOF. Note that the aggregated process has a finite mean (by assumption) and finite variance, which for any $H \in (1/2, 1)$ is

$$Var[X_t] = \sum_{k=1}^{\infty} \sigma^2_{X_k} = \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} < \infty. \tag{5.3}$$

This implies that X_t is a well defined process in the L^2 sense. We next prove the long-memory of the process by showing that the aggregated process has an autocovariance function of the form $R(u) \simeq A_1(u)u^{-\tau}$ with $\tau \in (0, 1)$ as $u \to \infty$ and spectral density of the form $f(\omega) \simeq A_2(\omega)|\omega|^{-\kappa}$ with $\kappa \in (0, 1)$ as $\omega \to 0$, where both A_1 and A_2 are slowly varying functions at infinity and at zero respectively.

Autocovariance function

Let $R^{(k)}(u) = \sum_{k=1}^{\infty} Cov(X_t^{(k)}, X_{t-u}^{(k)})$ represent the autocovariance function of the individual INAR(1) processes, then under the conditions of (5.2) we obtain that the covariance of the aggregated process at lag $u = t - s$ is given by

$$\begin{aligned}
R(u) &= \sum_{k=1}^{\infty} R^{(k)}(u) = \sum_{k=1}^{\infty} \sigma^2_{X_k}\, \alpha_k^{|u|} = \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \; \exp\{-|u|c_2/k\} \\
&\simeq \int_1^{\infty} \frac{c_1}{x^{1+2(1-H)}} \; \exp\{-|u|c_2/x\}\, dx = \frac{c_1}{(|u|c_2)^{2(1-H)}} \int_0^{|u|c_2} z^{2(1-H)-1} e^{-z} dz \\
&\simeq \frac{c_1 \Gamma(2(1-H))}{(|u|c_2)^{2(1-H)}} = \left(\frac{c_1 \Gamma(2(1-H))}{c_2^{2(1-H)}} \right) \frac{1}{|u|^{2(1-H)}} \quad as \quad |u| \to \infty
\end{aligned} \tag{5.4}$$

where $u \in \mathbb{Z}$ and $H \in (0, 1)$. Note that a substitution of $z = |u|\, c_2 \,/\, x$ was made to the integral in the third line of the proof. If $H \in (1/2, 1)$ then equation (5.4) satisfies the definition of long-memory above.

Spectral density

Note that Barndorff-Nielsen in [1] constructed a long-memory process with the same autocovariance function $R(u)$ as (5.4) but in continuous time so that $u \in \mathbb{R}$ (for an extension of these results see [5, 25]). The corresponding spectral density $\tilde{f}(\omega)$, $\omega \in \mathbb{R}$, therefore exists (see e.g. [26, pp. 210–226]) and may be obtained directly from the autocovariance function (5.4). The identity $f(\omega) = \sum_{s=-\infty}^{\infty} \tilde{f}(\omega + 2\pi s)$ where $-\pi \leq \omega < \pi$ may then be used to find the spectral density of the discrete time process with autocovariance structure of the form (5.4).

Let $\tilde{f}(\omega)$ denote the spectral density of a continuous time process $\{X_t; t \in \mathbb{R}\}$, then the spectral density for a process with autocovariance function of the form (5.4) under the conditions of proposition (5.1) is derived on re-writing $R(u)$ as

$$\begin{aligned} R(u) &= \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \exp\{-|u|c_2/k\} = \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{kc_2}{c_2^2 + k^2 w^2} e^{i\omega u} d\omega \\ &= R(0) \int_{-\infty}^{\infty} \left(\frac{1}{R(0)} \sum_{k=1}^{\infty} \frac{c_1 c_2}{\pi} \frac{1}{k^{2(1-H)}} \frac{1}{c_2^2 + \omega^2 k^2} \right) e^{i\omega u} d\omega = R(0) \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega u} d\omega. \end{aligned}$$

Hence the spectral density of the aggregated process in continuous time with autocovariance function $R(u), u \in \mathbb{R}$ of the form (5.4) has spectral density

$$\tilde{f}(\omega) = \frac{1}{\sigma_X^2} \sum_{k=1}^{\infty} \frac{c_1 c_2}{\pi} \frac{1}{k^{2(1-H)}} \frac{1}{c_2^2 + \omega^2 k^2}, \quad \omega \in \mathbb{R}.$$

The equivalent spectral density for the discrete time process $f(\omega)$ $(-\pi \leq \omega < \pi)$ is therefore

$$f(\omega) = \sum_{s=-\infty}^{\infty} \tilde{f}(\omega + 2\pi s) = \frac{1}{\sigma_X^2} \frac{c_1 c_2}{\pi} \sum_{s=-\infty}^{\infty} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2(1-H)}} \frac{1}{c_2^2 + (w + 2\pi s)^2 k^2} \right]. \quad (5.5)$$

Note that the spectral density has a pole at the origin ($\omega = 0$) for $H \in (1/2, 1)$. The term in expression (5.5), for $H \in (1/2, 1)$, corresponding to $s = 0$ and $\omega = 0$ is

$$\frac{1}{\sigma_X^2} \frac{c_1}{c_2 \pi} \sum_{k=1}^{\infty} \frac{1}{k^{2(1-H)}} = \infty, \qquad c_1,\ c_2 > 0,$$

and the remaining terms in the summation are strictly positive; we therefore have $f(0) = \infty$. The spectral density can be simplified by interchanging the summation which gives

$$\begin{aligned} f(\omega) &= \frac{1}{\sigma_X^2} \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \frac{1}{2\pi} \frac{1 - \exp\{-2c_2/k\}}{1 - 2\exp\{-c_2/k\}\cos\omega + \exp\{-2c_2/k\}} \\ &= \frac{1}{\sigma_X^2} \sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \frac{1}{2\pi} \frac{\cosh(c_2/2k)\sinh(c_2/2k)}{\cosh^2(c_2/2k) - \cos^2(w/2)}, \quad -\pi \leq \omega < \pi. \qquad \blacksquare \end{aligned}$$

Long-memory process with a Poisson marginal distribution

We now construct a stationary long-memory process with a Poisson ($\text{Po}(\lambda)$) marginal distribution with autocovariance function of the form (5.4) and spectral density of the form (5.5) by the aggregation of independent Poisson INAR(1) processes.

Proposition 5.2 *Let $\{X_t^{(k)}; k = 1, 2, \ldots\}$ be a sequence of stationary and independent $\text{Po}(\lambda_k)$ INAR(1) processes with thinning parameter $\alpha_k = \exp\{-c_2/k\}$ $(c_2 > 0)$ where*

$$\lambda_k = \frac{\lambda}{\zeta(1+2(1-H))} \frac{1}{k^{1+2(1-H)}}, \quad H \in (1/2, 1)$$

and $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ is the Riemann-Zeta function. Then the aggregated process $X_t = \sum_{k=1}^{\infty} X_t^{(k)}$ has long-memory with Hurst parameter H and a $\text{Po}(\lambda)$ marginal distribution with autocovariance function

$$R(u) = \frac{\lambda}{\zeta(1+2(1-H))} \sum_{k=1}^{\infty} \frac{\exp\{-c_2|u|/k\}}{k^{1+2(1-H)}}, \quad u \in \mathbb{Z}$$

and spectral density

$$f(\omega) = \frac{c_2}{\pi\zeta(1+2(1-H))} \sum_{s=-\infty}^{\infty} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2(1-H)}} \frac{1}{c_2^2 + (w+2\pi s)^2 k^2} \right], \quad -\pi \leq \omega < \pi.$$

PROOF. As the Poisson distribution is discrete infinitely-divisible, a Poisson random variable with mean λ can be represented as the infinite sum of Poisson random variables with mean λ_k with $\sum_{k=1}^{\infty} \lambda_k = \lambda$. Let us assume that the $X_t^{(k)}$ follow a Poisson $\text{Po}(\lambda_k)$ distribution and that X_t follows a Poisson $\text{Po}(\lambda)$ distribution.

For X_t to have long memory we require $\sigma_{X_k}^2 = \frac{c_1}{k^{1+2(1-H)}}$ (eq. (5.2)) and in the case of a Poisson INAR(1) process with deterministic thinning we have $\mu_{X_k}^2 = \sigma_{X_k}^2 = \lambda_k$. For any $t \in \mathbb{Z}$, using the fact $G_{X_t}(s) = \prod_{k=1}^{\infty} G_{X_t^{(k)}}(s)$, we therefore require that

$$\exp(\lambda(s-1)) = \exp\left(\sum_{k=1}^{\infty} \lambda_k (s-1) \right) = \exp\left(\left[\sum_{k=1}^{\infty} \frac{c_1}{k^{1+2(1-H)}} \right] (s-1) \right),$$

which implies that the constant c_1 and the parameter λ_k must be of the form

$$c_1 = \frac{\lambda}{\sum_{k=1}^{\infty} k^{-[1+2(1-H)]}} \Rightarrow \lambda_k = \lambda \left(\frac{k^{-[1+2(1-H)]}}{\sum_{k=1}^{\infty} k^{-[1+2(1-H)]}} \right). \qquad \blacksquare$$

It is clear from the form of λ_k that the aggregated long-memory process is a sum of weighted Poisson processes whose mean and variance tend to zero in the limit as $k \to \infty$.

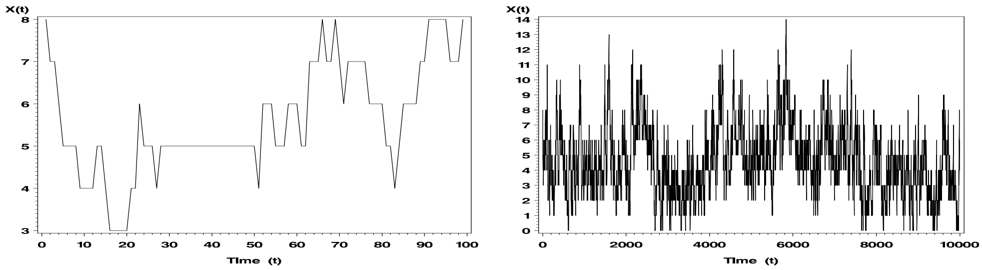

Figure 5.1: Realization of a long memory Poisson INAR(1) series

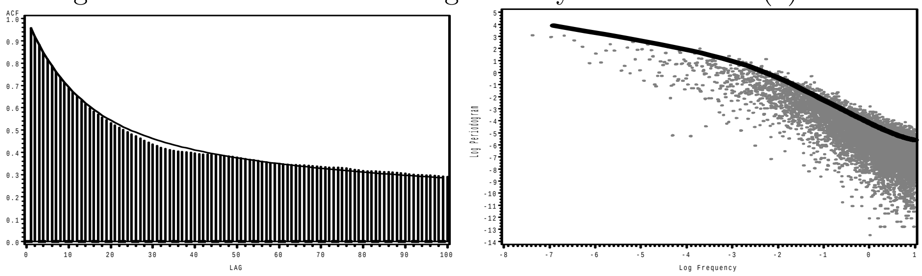

Figure 5.2: Autocorrelation function & periodogram

Figure 5.1 shows part of a realization of a simulated long-memory Poisson INAR(1) process of length $t = 20000$ with Hurst parameter $H = 0.8$, $\lambda = 5$ and constant $c_2 = 0.1$. Note that c_1 is specified by the parameters of the marginal distribution of the long-memory process whereas c_2 is a free parameter. The simulations show both the short term ($t = 1000$) and long term ($t = 10000$) behaviour of the process. Figure 5.2 shows the autocorrelation function and periodogram in logarithmic scale of the simulated long-memory process, with the solid line showing the theoretical value of the autocorrelation function and spectral density respectively.

Long-memory process with a NBD marginal distribution

We now construct a stationary long-memory process with a negative binomial marginal distribution ($\mathrm{NBD}^{(\gamma,\beta)}$) with autocovariance function of the form (5.4) and spectral density of the form (5.5) by the aggregation of independent NBD INAR(1) processes.

Proposition 5.3 *Let* $\{X_t^{(k)}; k = 1, 2, \ldots\}$ *be a sequence of stationary and independent* $\mathrm{NBD}^{(\gamma_k,\beta)}$ *INAR(1) processes with thinning parameter* $\alpha_k = \exp\{-c_2/k\}$ $(c_2 > 0)$. *Additionally, let* γ_k *be of the form*

$$\gamma_k = \frac{\gamma}{\zeta(1 + 2(1 - H))} \frac{1}{k^{1+2(1-H)}}, \quad H \in (1/2, 1)$$

where $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ *is the Riemann-Zeta function. Then the aggregated process* $X_t = \sum_{k=1}^{\infty} X_t^{(k)}$ *has long-memory with Hurst parameter* H *and a* $\mathrm{NBD}^{(\gamma,\beta)}$ *marginal* *distribution with covariance function*

$$R(u) = \frac{\gamma}{\zeta(1+2(1-H))}\left(\frac{\beta+1}{\beta^2}\right)\sum_{k=1}^{\infty}\frac{\exp\{-c_2|u|/k\}}{k^{1+2(1-H)}}, \quad u \in \mathbb{Z}$$

and spectral density

$$f(\omega) = \frac{c_2}{\pi\zeta(1+2(1-H))}\sum_{s=-\infty}^{\infty}\left[\sum_{k=1}^{\infty}\frac{1}{k^{2(1-H)}}\frac{1}{c_2^2+(w+2\pi s)^2k^2}\right], \quad -\pi \le \omega < \pi.$$

PROOF. As the NBD is discrete infinitely-divisible, a NBD$^{(\gamma,\beta)}$ random variable can be represented as the infinite sum of NBD$^{(\gamma_k,\beta)}$ random variables with $\sum_{k=1}^{\infty}\gamma_k = \gamma$. Let us assume that the $X_t^{(k)}$ follow a NBD$^{(\gamma_k,\beta)}$ distribution and that X_t follows a NBD$^{(\gamma,\beta)}$ distribution.

For X_t to have long memory we require $\sigma^2_{X_k} = \frac{c_1}{k^{1+2(1-H)}}$ (eq. (5.2)). We have

$$\sigma^2_{X_t^{(k)}} = \gamma_k\left(\frac{\beta+1}{\beta^2}\right) = \frac{c_1}{k^{1+2(1-H)}} \Rightarrow \gamma_k = \left(\frac{\beta+1}{\beta^2}\right)^{-1}\frac{c_1}{k^{1+2(1-H)}}.$$

For any $t \in \mathbb{Z}$, using the fact $G_{X_t}(s) = \prod_{k=1}^{\infty} G_{X_t^{(k)}}(s)$, we also require

$$\left(\frac{\beta+1-s}{\beta}\right)^{-\gamma} = \left(\frac{\beta+1-s}{\beta}\right)^{-\sum_{k=1}^{\infty}\gamma_k} \Rightarrow \gamma = \sum_{k=1}^{\infty}\gamma_k = \sum_{k=1}^{\infty}\left(\frac{\beta+1}{\beta^2}\right)^{-1}\frac{c_1}{k^{1+2(1-H)}}$$

$$\Rightarrow \gamma_k = \gamma\left(\frac{k^{-[1+2(1-H)]}}{\sum_{k=1}^{\infty}k^{-[1+2(1-H)]}}\right). \quad \blacksquare$$

The aggregated long-memory process, using the form of γ_k, is a sum of weighted NBD processes whose mean and variance tend to zero in the limit as $k \to \infty$.

Figure 5.3 shows part of a realization of a simulated long-memory NBD INAR(1) process of length $t = 20000$ with Hurst parameter $H = 0.8$, $\gamma = 5, \beta = 1$ and constant $c_2 = 0.1$. Note that the value of the constant c_1 is specified by parameters of the marginal distribution of the long-memory process whereas the constant c_2 is a free parameter. The simulations show both the short term ($t = 1000$) and long term ($t = 10000$) behaviour of the process. Figure 5.4 shows the autocorrelation function and periodogram in logarithmic scale of the simulated long-memory process, with the solid line showing the theoretical value of the autocorrelation function and spectral density respectively.

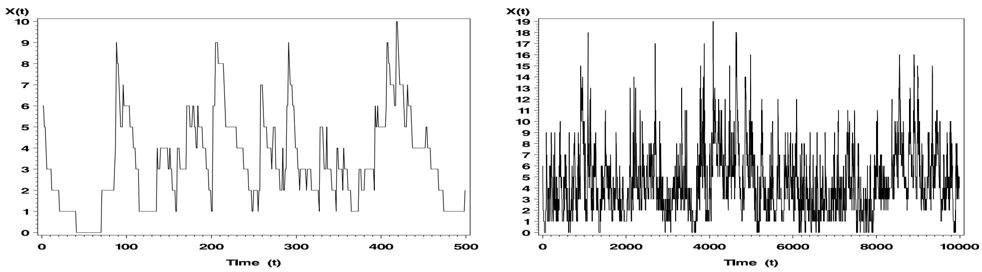

Figure 5.3: Long Memory NBD INAR(1) Realization

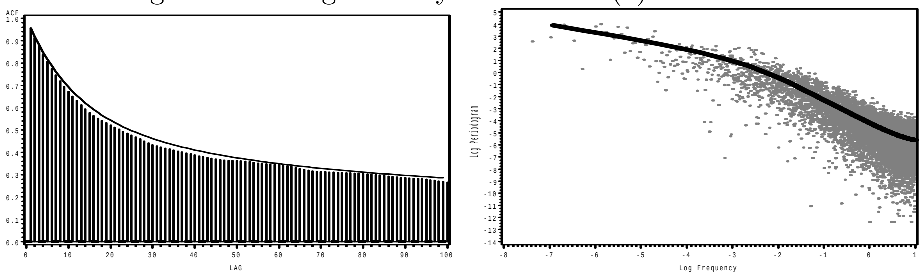

Figure 5.4: Autocorrelation Function & Periodogram

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