# Entropies of the partitions of the unit interval generated by the Farey tree

Nikolai Moshchevitin[1] and Anatoly Zhigljavsky[2]

**Abstract**

Let $\mathcal{F}_n$ be the set of fractions $p/q \in [0,1]$ such that the sum of partial quotients in their continued fraction representation is not greater than $n$. We consider the sum $\sigma_\beta(\mathcal{F}_n) = \sum (qq')^{-\beta}$ taken over the denominators of neighbouring fractions in $\mathcal{F}_n$ and prove that for all $\beta > 1$

$$\sigma_\beta(\mathcal{F}_n) = \frac{2}{n^\beta}\frac{\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{\log n}{n^{(\beta+1)(2\beta-1)/(2\beta)}}\right) \quad \text{as} \quad n \to \infty\,,$$

where $\zeta(\cdot)$ is the Riemann zeta-function.

**Key words:** Farey tree, continued fractions, transfer operator, Farey map, partition of the interval.

**AMS classification:** 11J70, 11K60, 11D68.

## 1 Introduction

The Farey tree $\mathcal{F}$ is an infinite binary tree whose nodes are labeled by rationals in $[0,1]$; it can be defined inductively as follows.

Consider the two-point set $\mathcal{F}_1 = \{0,1\}$ with 0 and 1 written as $\frac{0}{1}$ and $\frac{1}{1}$ respectively. Let $n \geq 1$ and $0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1$ be the fractions in $\mathcal{F}_n$ arranged in order of increase and written in lowest terms; here $N(n) = 2^{n-1}$. Then

$$\mathcal{F}_{n+1} = \mathcal{F}_n \bigcup \mathcal{Q}_{n+1} \quad \text{with} \quad \mathcal{Q}_{n+1} = \{x_{i-1,n} \oplus x_{i,n}, \;\; i = 1,\ldots,2^{n-1}\}\,,$$

[1]Faculty of Mechanics and Mathematics, Moscow State University, 119899 Moscow, Russia, e-mail moshche@mech.math.msu.su; the work was partly supported by Russian Science Foundation grants RFFI 02-01-00912, NSh-136.2003.1 and MD-3321.2004.1 as well as INTAS grant 03-51-5070

[2]*The author for correspondence;* Mathematics Institute, Cardiff University, Cardiff CF24 4YH, UK, e-mail: ZhigljavskyAA@cardiff.ac.uk

where

$$\frac{p}{q} \oplus \frac{p'}{q'} = \frac{p+p'}{q+q'}$$

is the mediant of the fractions $\frac{p}{q}$ and $\frac{p'}{q'}$. For example,

$$\mathcal{F}_2 = \{0, \tfrac{1}{2}, 1\}, \;\; \mathcal{F}_3 = \{0, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, 1\} \;\text{ and }\; \mathcal{F}_4 = \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, 1\}\,.$$

$\mathcal{F}_n$ is sometimes called Brocot sequence of order $n$. Elements of $\mathcal{Q}_n$ are known as Farey fractions of level $n$; for $n \geq 2$ they are the nodes in the Farey tree at level $n-1$. The first (root) node of the tree is $\frac{1}{2}$.

It is well known (see e.g. Schroeder (1991), p. 337) and straightforward to check that for the Farey fractions of level $n$ the sum of partial quotients in their continued fraction representation is exactly $n$; that is,

$$\mathcal{Q}_n = \{\tfrac{p}{q} = 1/(a_1{+}1/(a_2{+}...{+}1/a_t)...) \text{ with } a_t{\geq}2 \text{ and } a_1{+}\ldots{+}a_t{=}n\}\,.$$

Fractions $x_{i,n} \in \mathcal{F}_n$ considered as points in the interval $[0,1]$ make a partition of this interval into $2^{n-1}$ subintervals of different length varying from $1/(F_nF_{n+1})$ to $1/n$, where $F_k$ is the $k$-th Fibonacci number. This partition is obviously non-uniform; in this paper we study the asymptotic behaviour of a natural characteristic that measures this non-uniformity and is defined as follows.

Let $0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1$ be some points in $[0,1]$ and $p_{i,n} = x_{i,n} - x_{i-1,n}$ $(i = 1, \ldots, N(n))$ be the lengths of the subintervals $[x_{i-1,n}, x_{i,n})$. For a fixed $\beta$ we set

$$\sigma_\beta^{(n)} = \sigma(x_{0,n}, \ldots, x_{N(n),n}) = \sum_{i=1}^{N(n)} p_{i,n}^\beta\,. \tag{1}$$

$\sigma_\beta^{(n)}$ is one of the most widely used characteristics of the uniformity of the partition of $[0,1]$ generated be the points $x_{i,n}$, see e.g. Drobot (1981). The value $\frac{1}{1-\beta}\log\sigma_\beta^{(n)}$ is the Rényi entropy of order $\beta \neq 1$ of this partition (as $\beta \to 1$ the Rényi entropies tend to the Shannon entropy of the partition). If the partition is defined by a dynamical system, then the properly normalised sequence of $\sigma_\beta^{(n)}$ converges to the maximum eigenvalue of the transfer operator (3), see e.g. Vallée B. (2001). In an important special case $\beta = 2$ the quantity (1) can be interpreted as the average length of the interval

$[x_{i-1,n}, x_{i,n})$ which a random uniformly distributed point in $[0,1]$ falls in; see Section 4.2 in Pronzato et al (1999) for details.

In a number of papers including Hall (1970) and Kanemitsu *et al* (1982), the limiting behaviour of $\sigma_\beta^{(n)}$ is studied in the case when $\{x_{i,n}\}$ is the Farey sequence of order $n$; that is, the set of all fractions $p/q$ in $[0,1]$ with $\gcd(p,q) = 1$ and $q \le n$. In the present paper we study the limiting behaviour of $\sigma_\beta^{(n)}$ when $x_{i,n}$ are the elements of $\mathcal{F}_n$; in this case we will write $\sigma_\beta^{(n)} = \sigma_\beta(\mathcal{F}_n)$.

The expression for $\sigma_\beta(\mathcal{F}_n)$ can be simplified using the fact that if $\frac{p}{q}$ and $\frac{p'}{q'}$ are two neighbours in $\mathcal{F}_n$ such that $\frac{p}{q} < \frac{p'}{q'}$, then the length of the interval $[\frac{p}{q}, \frac{p'}{q'})$ is $p'/q' - p/q = 1/(qq')$. This yields

$$\sigma_\beta(\mathcal{F}_n) = \sum_{(q,q')} \frac{1}{(qq')^\beta} ,$$

where the sum is taken over the set of pairs of denominators of all the neighbours in $\mathcal{F}_n$.

The following theorem contains the main result of this paper.

**Theorem 1.** *For any $\beta > 1$ we have*

$$\sigma_\beta(\mathcal{F}_n) = \frac{2}{n^\beta} \frac{\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{\log(n)}{n^{(\beta+1)(2\beta-1)/(2\beta)}}\right) \quad \text{as } n \to \infty . \tag{2}$$

Proof of the theorem is given in Section 3.

Note that for small $\beta > 1$ the rate of convergence in (2) is slow. However, the fact that for all $\beta > 1$ the main term for $\sigma_\beta(\mathcal{F}_n)$ is indeed $\frac{2}{n^\beta}\frac{\zeta(2\beta-1)}{\zeta(2\beta)}$ agrees with numerical experiments that the authors have been carried out. As an example, Table 1 illustrates the rate of convergence in (2) for $\beta = 2$, where the main term in the asymptotical formula (2) is $\frac{2\zeta(3)}{\zeta(4)} n^{-2} \cong 2.22125 n^{-2}$.

The order of decrease for $\sigma_\beta(\mathcal{F}_n)$ as $n \to \infty$ is different from $n^{-\beta}$ for $\beta \le 1$. Thus, $\sigma_0(\mathcal{F}_n) = 2^{n-1}$ (it is the number of intervals in the $n$-th level partition) and $\sigma_1(\mathcal{F}_n) = 1$, see Lemma 2.

| $n$ | $\tilde{\sigma}_n$ | $n$ | $\tilde{\sigma}_n$ | $n$ | $\tilde{\sigma}_n$ | $n$ | $\tilde{\sigma}_n$ |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 12 | 2.225125138 | 22 | 2.198914837 | 32 | 2.199080361 |
| 3 | 1.666666667 | 13 | 2.218892938 | 23 | 2.198552286 | 33 | 2.199321142 |
| 4 | 2.013333333 | 14 | 2.213886731 | 24 | 2.198328443 | 34 | 2.199573834 |
| 5 | 2.172902494 | 15 | 2.209932920 | 25 | 2.198214857 | 35 | 2.199835181 |
| 6 | 2.237347594 | 16 | 2.206846310 | 26 | 2.198188931 | 36 | 2.200102532 |
| 7 | 2.257088762 | 17 | 2.204459272 | 27 | 2.198232662 | 37 | 2.200373722 |
| 8 | 2.257418374 | 18 | 2.202630388 | 28 | 2.198331652 | 38 | 2.200646994 |
| 9 | 2.250363055 | 19 | 2.201244461 | 29 | 2.198474352 | 39 | 2.200920916 |
| 10 | 2.241355171 | 20 | 2.200209351 | 30 | 2.198651468 | 40 | 2.201194325 |
| 11 | 2.232670354 | 21 | 2.199452029 | 31 | 2.198855495 | 41 | 2.201466280 |

Table 1: Numerically computed values of $\tilde{\sigma}_n = n(n-1)\sigma_2(\mathcal{F}_n)$ for $n = 2, \ldots, 41$.

# 2 Reformulation of the problem in terms of dynamical systems and continued fractions

## 2.1 Reformulation in terms of dynamical systems

The Farey map $T: [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} x/(1-x) & \text{if } 0 \le x < 1/2 \\ (1-x)/x & \text{if } 1/2 \le x \le 1\,. \end{cases}$$

The map is shown in Figure 1.

There is a simple relation between the Farey fractions of level $n$ and the Farey map:

$$\mathcal{Q}_n = T^{-n+1}(1) = \{x \in [0,1] \text{ such that } T^{n-1}(x) = 1\} \;\; \forall n \ge 2$$

implying $\mathcal{F}_n = T^{-n}(0)$ for all $n \ge 1$.

The Farey map belongs to a class of the so-called almost expanding maps. It has the absolutely continuous invariant density $p(x) = 1/x$ $(0 < x < 1)$ and it is ergodic with respect to this density; the density $p(x)$ has infinite mass implying that the metric entropy of $T(\cdot)$ is zero (for details see e.g. Lagarias (1992)). Moreover, the topological pressure $P_\beta$ of the Farey map

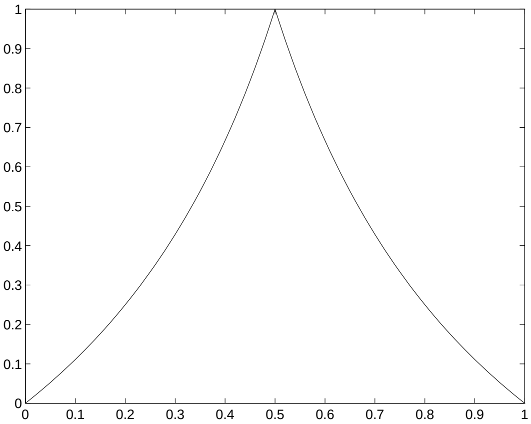


Figure 1: The Farey map

is zero for $\beta \geq 1$, see Prellberg and Slawny (1992). The pressure can be defined as $P_\beta = \log \lambda_\beta$, where $\lambda_\beta$ is the maximal eigenvalue of the transfer operator $\mathcal{L}_\beta : C[0,1] \to C[0,1]$ defined for $f \in C[0,1]$ by

$$\mathcal{L}_\beta f(x) = \sum_{y\colon T(y)=x} \frac{f(y)}{|T'(y)|^\beta} \,. \tag{3}$$

For the Farey map the pressure is

$$P_\beta = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(q,q')} \frac{1}{(qq')^\beta} \,, \tag{4}$$

where for fixed $n$ the sum is taken over the set of pairs of denominators of all the neighbours in $\mathcal{F}_n$.

Prellberg and Slawny (1992) studied the behaviour of the pressure $P_\beta$ for a class of almost expanding maps (including the Farey map) as $\beta \uparrow 1$; we consider a version of the pressure for $\beta > 1$. Theorem 1 shows that to obtain non-trivial limits, the normalisation of the sum in (4) with $\beta > 1$ must be different from $\frac{1}{n}\log$; this normalization is $n^\beta$. Similar phenomena seems to hold for some other almost expanding maps; this phenomena is related to the non-exponential divergence of the trajectories $x_{n+1} = T(x_n)$ for these maps.

## 2.2 Reformulation in terms of continued fractions

Let $\mathcal{A}$ be the set of all integer vectors $a = (a_1, ..., a_t)$ with $t \geq 1$, $a_j \geq 1$ $(j = 1, \ldots, t-1)$ and $a_t > 1$. Let also

$$\mathcal{A}_n = \{a = (a_1, \ldots, a_t) \in \mathcal{A} \text{ such that } a_1 + \ldots + a_t = n\} .$$

With each $a = (a_1, ..., a_t) \in \mathcal{A}$ we associate the continued fraction $1/(a_1 + 1/(a_2 + ... + 1/a_t)...)$ and the corresponding continuant (the denominator of the fraction), which we write as $[a_1, \ldots, a_t]$; empty continuant is equal to 1.

By construction, for all $n > 1$, each fraction in $\mathcal{F}_n \setminus (\mathcal{F}_1 \cup Q_n)$ has two neighbours which belong to the set $\mathcal{Q}_n$. Also, every fraction $p/q \in \mathcal{Q}_n$ has two neighbours, say $p_-/q_-$ and $p_+/q_+$, in $\mathcal{F}_n \setminus \mathcal{Q}_n$. Explicit formulae for the continuants of these neighbours are given below.

**Lemma 1.** *For each $a \in \mathcal{A}_n$, the fraction $p/q \in \mathcal{Q}_n$ with continuant $q = q(a) = [a_1, ..., a_t]$ has two neighbours in $\mathcal{F}_n$ with continuants*

$$q_- = q_-(a) = [a_1, ..., a_{t-1}] \text{ and } q_+ = q_+(a) = [a_1, ..., a_t - 1] . \tag{5}$$

*Similarly, any fraction $p/q \in \mathcal{F}_{n-1} \setminus \mathcal{F}_1$ with continuant $q = q(a) = [a_1, ..., a_t]$ has two neighbours in $\mathcal{F}_n$ with continuants*

$$[a_1, ..., a_t, n - (a_1 + ... + a_t)] \text{ and } [a_1, ..., a_t - 1, 1, n - (a_1 + ... + a_t)] . \tag{6}$$

The proof is a simple induction with respect to $n$. □

Note that the two neighbours of the fraction $p/q$ with continuants $q_-$ and $q_+$ are not simply left and right: the larger denominator $q_+$ can be on either side of $q$.

The first part of Lemma 1 implies that we can rewrite $\sigma_\beta(\mathcal{F}_n)$, the characteristic of interest, as

$$\sigma_\beta(\mathcal{F}_n) = \sum_{a \in \mathcal{A}_n} \left( \frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) , \tag{7}$$

where $q = q(a) = [a_1, ..., a_t]$ and $q_- = q_-(a)$, $q_+ = q_+(a)$ are defined (5).

For any $\beta > 1$ we can easily establish a lower bound for $\sigma_\beta(\mathcal{F}_n)$ in the following way:

$$\sigma_\beta(\mathcal{F}_n) = \sum_{a \in \mathcal{A}_n} \left( \frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) \geq \sum_{a \in \mathcal{A}_n} \frac{1}{(qq_-)^\beta} \geq \frac{1}{n^\beta} \sum_{a \in \mathcal{A}_n} \frac{1}{(q_-)^{2\beta}}$$

$$\geq \frac{1}{n^\beta} \sum_{a \in \mathcal{A}_n:\, q_- < n} \frac{1}{(q_-)^{2\beta}} = \frac{2}{n^\beta} \sum_{q_-=1}^{n-1} \frac{\varphi(q_-)}{(q_-)^{2\beta}} = \frac{2}{n^\beta} \sum_{l=1}^{\infty} \frac{\varphi(l)}{l^{2\beta}} + O\left(\frac{1}{n^{2\beta-1}}\right) \quad \text{as } n \to \infty\,.$$

In deriving the lower bound we have neglected the terms with large denominators $q_-(a)$, used the inequality $q \leq nq_-$, see (10), and used the fact that in the partition generated by $\mathcal{F}_n$ there are exactly $2\varphi(l)$ intervals with $q_-(a) = l < n$ as one of their end-points.

Theorem 1 states that the just derived lower bound

$$\liminf_{n\to\infty} n^\beta \sigma_\beta(\mathcal{F}_n) \geq 2 \sum_{l=1}^{\infty} \frac{\varphi(l)}{l^{2\beta}} = \frac{2\zeta(2\beta-1)}{\zeta(2\beta)}$$

is in reality the exact limit of $n^\beta \sigma_\beta(\mathcal{F}_n)$ as $n \to \infty$. Thus, the main contribution to the sum $\sigma_\beta(\mathcal{F}_n)$ is made by a few terms with small denominators $q_-$ ('major arcs').

# 3 Proof of Theorem 1

## 3.1 More notation and the main steps in the proof

We assume throughout this section that $\beta > 1$ is a fixed number. Set

$$r = r(\beta) = \frac{3\beta - 2}{2(\beta-1)} \quad \text{and} \quad w = w(n,\beta) = \min\{n/2, n^{(\beta+1)/(2\beta)} \log n\ \}\,. \tag{8}$$

The value of $r$ is chosen to satisfy $(\beta-1)(2r-1) = 2\beta-1$, see Lemma 3; $w = w(n,\beta)$ will minimize the error term. Note that since $\beta > 1$ we have $w = o(n)$ as $n \to \infty$.

For a subset $\mathcal{A}_{n,i}^{(j)}$ of $\mathcal{A}$, denote

$$\Sigma_{n,i}^{(j)} = \sum_{a \in \mathcal{A}_{n,i}^{(j)}} \left( \frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right),$$

where $a = (a_1, ..., a_t)$, $q = q(a) = [a_1, ..., a_t]$; $q_- = q_-(a)$ and $q_+ = q_+(a)$ are defined in (5).

The proof of Theorem 1 is based on a few lemmas and uses the splitting of the sum $\sigma_\beta(\mathcal{F}_n)$, which is the sum over $\mathcal{A}_n$, into the sums $\Sigma_{n,i}^{(j)}$ over smaller subsets of indices $a$. The first split of $\mathcal{A}_n$ is into the following two subsets:

$$\mathcal{A}_{n,1}^{(1)} = \{a = (a_1, \ldots, a_t) \in \mathcal{A}_n \text{ such that } [a_1, \ldots, a_t] < n^r\} \quad \text{and}$$

$$\mathcal{A}_{n,2}^{(1)} = \mathcal{A}_n \setminus \mathcal{A}_{n,1}^{(1)} = \{a = (a_1, \ldots, a_t) \in \mathcal{A}_n \text{ such that } [a_1, \ldots, a_t] \geq n^r\}\,.$$

We then split the index set $\mathcal{A}_{n,1}^{(1)}$ into two subsets as follows:

$$\mathcal{A}_{n,1}^{(2)} = \{a \in \mathcal{A}_{n,1}^{(1)} \text{ such that } \max_{1\leq j\leq t} a_j > n - w\}\,,$$

$$\mathcal{A}_{n,2}^{(2)} = \mathcal{A}_{n,1}^{(1)} \setminus \mathcal{A}_{n,1}^{(2)} = \{a \in \mathcal{A}_{n,1}^{(1)} \text{ such that } \max_{1\leq j\leq t} a_j \leq n - w\}\,.$$

Thus, all $a \in \mathcal{A}_{n,1}^{(2)}$ have at least one very large partial quotient $a_j$; on the other hand, all $a_j$'s for $a \in \mathcal{A}_{n,2}^{(2)}$ are relatively small.

Next we split the set $\mathcal{A}_{n,1}^{(2)}$ into the set where the largest partial quotient is the last one and where it is not:

$$\mathcal{A}_{n,1}^{(3)} = \Big\{a = (a_1, \ldots, a_t) \in \mathcal{A}_{n,1}^{(2)} \text{ such that } a_t > \max\{a_1, ..., a_{t-1}\}\Big\} \quad \text{and}$$

$$\mathcal{A}_{n,2}^{(3)} = \mathcal{A}_{n,1}^{(2)} \backslash \mathcal{A}_{n,1}^{(3)} = \Big\{a = (a_1, ..., a_t) \in \mathcal{A}_{n,1}^{(2)} \text{ such that } a_t \leq \max\{a_1, ..., a_{t-1}\}\Big\}\,.$$

The split of the set $\mathcal{A}_n$ into subsets is shown in Figure 2.

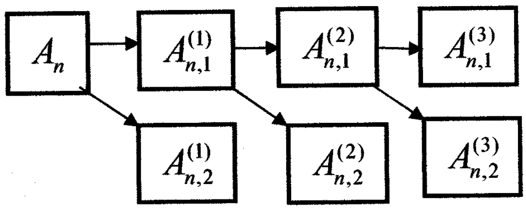


Figure 2: Split of the index set $\mathcal{A}_n$.

Additionally, we split the sum $\Sigma_{n,1}^{(3)}$ as

$$\Sigma_{n,1}^{(3)} = \Sigma_{n,1}^{(3)+} + \Sigma_{n,1}^{(3)-} \quad \text{with } \Sigma_{n,1}^{(3)+} = \sum_{a\in\mathcal{A}_{n,1}^{(3)}} \frac{1}{(qq_+)^\beta} \quad \text{and } \Sigma_{n,1}^{(3)-} = \sum_{a\in\mathcal{A}_{n,1}^{(3)}} \frac{1}{(qq_-)^\beta}\,.$$

As a result, we have the following split of the sum $\sigma_\beta(\mathcal{F}_n)$ defined in (7):

$$\sigma_\beta(\mathcal{F}_n) = \Sigma_{n,2}^{(1)} + \Sigma_{n,2}^{(2)} + \Sigma_{n,2}^{(3)} + \Sigma_{n,1}^{(3)+} + \Sigma_{n,1}^{(3)-}\,. \tag{9}$$

In Lemma 3 we consider the sum $\Sigma_{n,2}^{(1)}$ (which accounts for all the terms with very large denominators) and establish that $\Sigma_{n,2}^{(1)} \le n^{-(2\beta-1)}$ for all $n$.

In Lemma 5 we prove that $\Sigma_{n,2}^{(2)} + \Sigma_{n,2}^{(3)} \ll (\log n)^{2\beta}/w^{2\beta-1}$ as $n \to \infty$.

In Lemma 6 we demonstrate that $\Sigma_{n,1}^{(3)+} \ll n^{-2\beta}$ as $n \to \infty$; this sum is obviously asymptotically dominated by the cumulative effect of the others. Finally, in Lemma 7 we prove that

$$\Sigma_{n,1}^{(3)-} = \frac{1}{n^\beta} \cdot \frac{2\zeta(2\beta-1)}{\zeta(2\beta)} + O\left(\frac{1}{n^{2\beta-1}} + \frac{w}{n^{\beta+1}} + \frac{1}{n^\beta w^{2(\beta-1)}}\right) \quad \text{as } n \to \infty.$$

Therefore, the decomposition (9) and Lemmas 3, 5, 6 and 7 imply the following asymptotic expansion for the sum $\sigma_\beta(\mathcal{F}_n)$ :

$$\sigma_\beta(\mathcal{F}_n) = \frac{2\zeta(2\beta-1)}{n^\beta \zeta(2\beta)} + O\left(\frac{1}{n^{2\beta-1}} + \frac{(\log n)^{2\beta}}{w^{2\beta-1}} + \frac{w}{n^{\beta+1}} + \frac{1}{n^\beta w^{2(\beta-1)}}\right) \quad \text{as } n \to \infty.$$

The choice of $w = w(n,\beta)$ in accordance with (8) provides the minimum error term which is $O\left(n^{-(\beta+1)(2\beta-1)/(2\beta)} \log(n)\right)$, as stated in (2). This result is an easy consequence of the fact that the sum of the second and the third terms in $O(\cdot)$ dominates the first and the forth terms. Note also that $(\beta+1)(2\beta-1)/(2\beta) > \beta$ for all $\beta > 1$ so that the error term is smaller than the main term in this range of $\beta$.

## 3.2 Lemmas

**Lemma 2.** *Let $n \ge 2$, $a = (a_1, ..., a_t) \in \mathcal{A}_n$, $q = q(a) = [a_1, ..., a_t]$, and $q_-$, $q_+$ be as defined in* (5). *We have:*

$$q = q_- + q_+ \le nq_- , \tag{10}$$

$$q_- \le q_+ \le a_t q_- , \tag{11}$$

$$\sum_{a \in \mathcal{A}_n} \left(\frac{1}{qq_-} + \frac{1}{qq_+}\right) = 1 \tag{12}$$

and

$$\frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \le \frac{n^{\beta-1}}{q^{2(\beta-1)}} \left(\frac{1}{qq_-} + \frac{1}{qq_+}\right). \tag{13}$$

**Proof.** The equality in (10) follows from the definition of $\mathcal{Q}_n$. For $n = 2$ the inequalities in (10) and (11) can be easily checked directly. For $n > 2$ these inequalities follow from (5) and the standard recurrence for the continuants of successive continued fraction convergents. The formula (12) expresses the fact that the total length of all the intervals in the partition generated by the points in $\mathcal{F}_n$ is 1.

Let us now prove (13). Using the inequalities $q_+ \geq q_-$ and $q_- \geq q/n$, see (10) and (11), for all $a \in \mathcal{A}_n$ we have:

$$\frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \leq \left(\frac{1}{qq_-} + \frac{1}{qq_+}\right) \max\left\{\frac{1}{(qq_-)^{\beta-1}}, \frac{1}{(qq_+)^{\beta-1}}\right\} =$$

$$\frac{1}{(qq_-)^{\beta-1}}\left(\frac{1}{qq_-} + \frac{1}{qq_+}\right) \leq \frac{n^{\beta-1}}{q^{2(\beta-1)}}\left(\frac{1}{qq_-} + \frac{1}{qq_+}\right).$$

□

**Lemma 3.** *For all $n \geq 1$ we have $\Sigma^{(1)}_{n,2} \leq n^{-(2\beta-1)}$.*

**Proof.** As $q \geq n^r$ for $a \in \mathcal{A}^{(1)}_{n,2}$, we obtain using the equality (12), the inequality (13) and the definition of $r$:

$$\Sigma^{(1)}_{n,2} \leq \frac{n^{\beta-1}}{q^{2(\beta-1)}} \sum_{a \in \mathcal{A}^{(1)}_{n,2}} \left(\frac{1}{qq_-} + \frac{1}{qq_+}\right) \leq \frac{n^{\beta-1}}{q^{2(\beta-1)}} \leq \frac{n^{\beta-1}}{n^{2r(\beta-1)}} = \frac{1}{n^{(\beta-1)(2r-1)}} = \frac{1}{n^{2\beta-1}}.$$

□

**Lemma 4.** *For all $a = (a_1, \ldots, a_t) \in \mathcal{A}^{(1)}_{n,1}$ with $n \geq 2$ we have $t \leq C \log n$, where $C = C(\beta) = r \log((\sqrt{5}+1)/2)$.*

**Proof** follows from the fact that for all $a = (a_1, \ldots, a_t) \in \mathcal{A}^{(1)}_{n,1}$ we have

$$\left(\frac{\sqrt{5}+1}{2}\right)^t \leq [a_1, ..., a_t] \leq n^r.$$

□

**Lemma 5.** *As $n \to \infty$, we have*

$$\Sigma^{(2)}_{n,2} + \Sigma^{(3)}_{n,2} \ll \frac{(\log n)^{2\beta}}{w^{2\beta-1}}.$$

**Proof.** Lemma 4 states that for all $a \in \mathcal{A}_{n,1}^{(1)}$ (that is, when $q(a) \le n^r$) we have $t \le C \log n$. As $n = a_1 + \ldots + a_t \le t \max a_j$, this implies $\max a_j \ge n/C \log n$.

Let $a = (a_1, ..., a_t) \in \mathcal{A}_{n,2}^{(2)}$ and $j$ be such that $a_j = \max\{a_1, ..., a_t\}$. Since $a_j \le n - w$, for the sum of remaining $a_i$'s we have $\sum_{i \ne j} a_i > w$ and similarly to the above, the second largest value of $a_i$'s is larger than or equal to $w/C \log n$. This implies that for any $a = (a_1, ..., a_t) \in \mathcal{A}_{n,2}^{(2)}$, there exist two indices $1 \le k \ne l \le t$ such that $a_k \ge w/C \log n$ and $a_l \ge w/C \log n$ and therefore for at least one index $j \le t-1$ we have $a_j \ge w/C \log n$.

If $a = (a_1, ..., a_t) \in \mathcal{A}_{n,2}^{(3)}$ there is $j < t$ such that $a_j = \max\{a_1, ..., a_t\}$. Since $\mathcal{A}_{n,2}^{(3)} \subseteq \mathcal{A}_{n,1}^{(2)}$, for this $a_j$ we have $a_j > n - w > w/\log n$ for all $n \ge 3$.

Set $c = \max\{1, 1/C\}$ and let $n \ge 3$. Then for all $a = (a_1, ..., a_t) \in \mathcal{A}_{n,2}^{(2)} \bigcup \mathcal{A}_{n,2}^{(3)}$ for at least one index $j \le t-1$ we have $a_j \ge cw/\log n$.

Therefore,

$$\Sigma_{n,2}^{(2)} + \Sigma_{n,2}^{(3)} \le 2 \sum_{\substack{a_1 + \ldots + a_{t-1} \le n, \\ \exists j \,:\, a_j \ge cw/\log n}} \left( \frac{1}{(qq_-)^\beta} + \frac{1}{(qq_+)^\beta} \right) \le 4 \sum_{\substack{a_1 + \ldots + a_{t-1} \le n, \\ \exists j \,:\, a_j \ge cw/\log n}} \frac{1}{(qq_-)^\beta}$$

(Here we have also used the fact that $q_+ \ge q_-$). Clearly,

$$q > a_j a_t \cdot [a_1, ..., a_{j-1}] \cdot [a_{j+1}, ..., a_{t-1}] \quad \text{and}$$

$$q_+ \ge q_- > a_j \cdot [a_1, ..., a_{j-1}] \cdot [a_{j+1}, ..., a_{t-1}] \, .$$

Hence $\Sigma_{n,2}^{(2)} + \Sigma_{n,2}^{(3)} \le$

$$4 \sum_{j < C \log n} \sum_{cw/\log n < a_j \le n} \sum_{\substack{a = (a_1, ..., a_t) \in \mathcal{A}_{n,1}^{(1)} : \\ a_j \text{ fixed; } j \le t-1}} \frac{1}{a_j^{2\beta} [a_1, ..., a_{j-1}]^{2\beta} \cdot [a_{j+1}, ..., a_{t-1}]^{2\beta} \cdot a_t^\beta}$$

$$\le \frac{4C \log n}{(cw/\log n)^{2\beta - 1}} \sum_{u+v \le w} \sum_{a_1 + \ldots + a_r = u} \frac{1}{[a_1, ..., a_r]^{2\beta}} \cdot \sum_{b_1 + \ldots + b_h = v} \frac{1}{[b_1, ..., b_{h-1}]^{2\beta} b_h^\beta}$$

$$\le \frac{4C (\log n)^{2\beta}}{(cw)^{2\beta - 1}} \cdot \sum_{a_1 + \ldots + a_r \le w} \frac{1}{[a_1, ..., a_r]^{2\beta}} \cdot \sum_{b_1 + \ldots + b_{h-1} \le w} \frac{1}{[b_1, ..., b_{h-1}]^{2\beta}} \sum_{b_h = 1}^{\infty} \frac{1}{b_h^\beta}$$

$$= \frac{4C\zeta(\beta)}{c^{2\beta - 1}} \frac{(\log n)^{2\beta}}{w^{2\beta - 1}} \cdot \left( \sum_{a_1 + \ldots + a_r \le w} \frac{1}{[a_1, ..., a_r]^{2\beta}} \right)^2$$

Since

$$\sum_{a_1+\ldots+a_r\le w}\frac{1}{[a_1,\ldots,a_r]^{2\beta}}\le\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^{2\beta}}=\frac{\zeta(2\beta-1)}{\zeta(2\beta)}$$

we obtain

$$\Sigma_{n,2}^{(2)}+\Sigma_{n,2}^{(3)}\le\frac{4C\zeta(\beta)}{c^{2\beta-1}}\left(\frac{\zeta(2\beta-1)}{\zeta(2\beta)}\right)^2\frac{(\log n)^{2\beta}}{w^{2\beta-1}}$$

and lemma follows. □

**Lemma 6.** *As $n\to\infty$, we have $\Sigma_{n,1}^{(3)+}\ll n^{-2\beta}$.*

**Proof.** Since $q_+<q\le nq_-=n\,[a_1,\ldots,a_{t-1}]$ for all $a=(a_1,\ldots,a_t)\in\mathcal{A}$, we have

$$\Sigma_{n,1}^{(3)+}=\sum_{\substack{a_1+\ldots+a_t=n,\\ a_t>n-w,\ q<n^r}}\frac{1}{(qq_+)^{\beta}}\le\sum_{n-w<a_t\le n}\ \sum_{a_1+\ldots+a_{t-1}=n-a_t}\frac{1}{(qq_+)^{\beta}}$$

$$\le\frac{1}{(n-w)^{2\beta}}\sum_{a_1+\ldots+a_{t-1}\le w}\frac{1}{[a_1,\ldots,a_{t-1}]^{2\beta}}\le\frac{2}{(n-w)^{2\beta}}\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^{2\beta}}\ll n^{-2\beta}\,.$$

□

**Lemma 7.** *As $n\to\infty$, we have*

$$\Sigma_{n,1}^{(3)-}=\frac{1}{n^{\beta}}\cdot\frac{2\zeta(2\beta-1)}{\zeta(2\beta)}+O\left(\frac{1}{n^{2\beta-1}}+\frac{w}{n^{\beta+1}}+\frac{1}{n^{\beta}w^{2(\beta-1)}}\right).$$

**Proof.**

$$\Sigma_{n,1}^{(3)-}=\sum_{\substack{a=(a_1,\ldots,a_t)\in\mathcal{A}_n,\\ q(a)<n^r,\ a_t>n-w}}\frac{1}{(q(a)q_-(a))^{\beta}}$$

$$=\sum_{\substack{a=(a_1,\ldots,a_t)\in\mathcal{A}_n,\\ a_t>n-w}}\frac{1}{(q(a)q_-(a))^{\beta}}-\sum_{\substack{a=(a_1,\ldots,a_t)\in\mathcal{A}_n,\\ q(a)\ge n^r,\ a_t>n-w}}\frac{1}{(q(a)q_-(a))^{\beta}}$$

The second sum is

$$\sum_{\substack{a_1+\ldots+a_t=n,\\ q(a)\ge n^r,\ a_t>n-w}}\frac{1}{(q(a)q_-(a))^{\beta}}=O\left(n^{-(2\beta-1)}\right)\quad\text{as } n\to\infty$$

and for the first sum we have

$$\sum_{\substack{a=(a_1,\ldots,a_t)\in\mathcal{A}_n,\\ a_t>n-w}} \frac{1}{(q(a)q_-(a))^\beta} = \sum_{\substack{a=(a_1,\ldots,a_t)\in\mathcal{A}_n,\\ a_1+\ldots+a_{t-1}\le w}} \frac{1}{(q(a)q_-(a))^\beta}\,.$$

In the Farey tree $\mathcal{F}_n$, each Farey fraction with denominator $q_-(a) = [a_1, ..., a_{t-1}]$ such that $a_1 + ... + a_{t-1} < n$ is a neighbour to two Farey fractions with denominators $q(a) = [a_1, ..., a_t]$ with $a_1 + ... + a_t = n$, see (6). Additionally, since

$$q = a_t q_- + (q_-)_- = q_-(n + O(w)) \ \text{ as } n \to \infty\,,$$

we have

$$\frac{1}{(q(a)q_-(a))^\beta} = \frac{1}{n^\beta q_-^{2\beta}}\left(1 + O\left(\frac{w}{n}\right)\right)\,.$$

In view of these two facts we obtain

$$\Sigma_{n,1}^{(3)-} = \frac{2}{n^\beta}\cdot \sum_{\substack{a_1+...+a_{t-1}\le w,\\ a_{t-1}>1 \text{ for } t>2}} \frac{1}{[a_1, ..., a_{t-1}]^{2\beta}} + O\left(n^{-(2\beta-1)} + \frac{w}{n^{\beta+1}}\right)$$

$$= \frac{2}{n^\beta}\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^{2\beta}} + O\left(\frac{1}{n^{2\beta-1}} + \frac{w}{n^{\beta+1}} + n^{-\beta}\cdot\sum_{a_1+...+a_{t-1}\ge w}\frac{1}{[a_1, ..., a_{t-1}]^{2\beta}}\right)\,.$$

As

$$\sum_{q=1}^{\infty}\frac{\varphi(q)}{q^{2\beta}} = \frac{\zeta(2\beta-1)}{\zeta(2\beta)} \quad \text{and}$$

$$\sum_{a_1+...+a_i\ge w}\frac{1}{[a_1, ..., a_i]^{2\beta}} \ll \sum_{q\ge w}\frac{1}{q^{2\beta-1}} \ll w^{-2(\beta-1)}\,,$$

lemma follows. □

# Acknowledgments

The authors are grateful to our collegues M.N. Huxley (Cardiff), P. Kargaev (St.Petersburg), L. Pronzato (Antibes) and H.P. Wynn (London) for useful discussions. The authors are also very grateful to the referee for very valuable comments.

# References


Drobot V. (1981) Uniform partitions of an interval. *Trans. Amer. Math. Soc.* 268, 151–160.

Hall R. R. (1970) A note on Farey series. *J. London Math. Society* (2), 2, 139–148.

Kanemitsu S., Sitaramachandra Rao R. and Siva Rama Sarma A. (1982) Some sums involving Farey fractions, *J. Math. Society of Japan*, 34, 125–142.

Lagarias J.C. (1992) Number theory and dynamical systems. *Proceedings of Symposia in Applied Mathematics,* **46**, 35-72.

Prellberg T. and Slawny J. (1992) Maps of intervals with indifferent fixed points: thermodynamic formalism and phase transitions. *J. Statist. Phys.* **66**, 503–514.

Pronzato L., Wynn H.P. and Zhigljavsky A.A. (1999) *Dynamical Search,* CRC / Chapman and Hall.

Schroeder M. (1991) *Fractals, chaos, power laws.* W.H. Freeman and Company, N.Y.

Vallée B. (2001) Dynamical sources in information theory: fundamental intervals and word prefixes. *Algorithmica* **29**, 262–306.