Kronecker Sequences: Asymptotic Distributions of the Partition Lengths

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1 Introduction: Statement of the Problem and Formulation of the Main Results

1.1 Kronecker sequences

Let θ be an irrational number in $[0,1)$, $x_k = k\theta \pmod 1$ for $k = 1, 2, \ldots$ and let $W_n(\theta) = \{x_1, \ldots, x_n\}$ be the Kronecker sequence of order n.

In the present work we derive asymptotic distributions of different characteristics associated with the interval lenghts of the partitions of [0,1] generated by $W_n(\theta)$. The main result establishes the two-dimensional asymptotic distribution of

$$(n \min\{x_1, \ldots, x_n\}, n(1 - \max\{x_1, \ldots, x_n\}))$$

when $n \to \infty$. It then yields a number of results concerning the asymptotic distributions of one-dimensional characteristics.

Assume that $y_{0,n} = 0$, $y_{n+1,n} = 1$ and $y_{k,n}$ $(k = 1, \ldots, n)$ be the members of $W_n(\theta)$ arranged in the order of increase. Define

$$\delta_n(\theta) = y_{1,n} = \min_{k=1,\ldots,n} x_k\,, \quad \Delta_n(\theta) = 1 - y_{n,n} = 1 - \max_{k=1,\ldots,n} x_k \tag{1}$$

and consider the partition of $[0,1)$ generated by $W_n(\theta)$:

$$\mathcal{P}_n(\theta) = \cup_{k=0}^{n} I_{k,n}\,, \quad \text{where } \; I_{k,n} = [y_{k,n}, y_{k+1,n})\,.$$

It is a well known property of the Kronecker sequence, see e.g. [3],[4], that for any $n \geq 1$ the partition $\mathcal{P}_n(\theta)$ of $[0,1)$ contains the intervals $I_{k,n}$ which lengths $|I_{k,n}|$ can only get two or three different values, namely, $\delta_n(\theta)$, $\Delta_n(\theta)$ and perhaps $\delta_n(\theta) + \Delta_n(\theta)$.

Denote

$$\alpha_n(\theta) = \min_{k=1,\ldots,n} |I_{k,n}| = \min\{\delta_n(\theta), \Delta_n(\theta)\}, \quad A_n(\theta) = \max_{k=1,\ldots,n} |I_{k,n}|\,,$$

$$\beta_n(\theta)=\max\{\delta_n(\theta),\Delta_n(\theta)\}\,,\quad \gamma_n(\theta)=\delta_n(\theta)+\Delta_n(\theta)\,,\quad \xi_n(\theta)=\frac{\alpha_n(\theta)}{\beta_n(\theta)}\,.$$

All these quantities, namely $\delta_n(\theta)$, $\Delta_n(\theta)$, $\alpha_n(\theta)$, $A_n(\theta)$, $\beta_n(\theta)$, $\gamma_n(\theta)$ and $\xi_n(\theta)$, give a rather complete description of the partition $\mathcal{P}_n(\theta)$. We are interested in their asymptotic behaviour, when $n \to \infty$. The main result of the paper is formulated in Theorem 1 below and presents the joint asymptotic distribution for $(n\delta_n(\theta), n\Delta_n(\theta))$. In Corollaries 1–4 and Theorem 2 we derive the one-dimensional asymptotic distributions for all charactersitics introduced above.

As demonstrated in Section 2, there is a close relation between the Kronecker and Farey sequences, and the quantities introduced above also characterize certain properties of the Farey sequences. (For example, $\alpha_n(\theta)$, which asymptotic distribution has been derived in [2], characterizes the error in approximation of θ by the Farey sequence of order n, see (11).) The present paper thus also studies some distributional properties of the Farey sequences.

In what follows 'meas' stands for the Lebesgue measure on $[0,1)$, $\{\cdot\}$ and $\lfloor\cdot\rfloor$ denote the fractional and integer part operations correspondingly, $\varphi(\cdot)$ is the Euler function and dilog $(\cdot)$ is the dilogarithm function: $\mathrm{dilog}(t)=\int_1^t \log s/(1-s)\mathrm{d}s$. Also, we shall say that a sequence of functions $\psi_n(\theta)$, $\theta \in [0,1)$, converges in distribution, when $n \to \infty$, to a probability measure with a density $q(\cdot)$ if for any $t>0$

$$\lim_{n\to\infty} \mathrm{meas}\{\theta \in [0,1) : \psi_n(\theta) \le t\} = \int_0^t q(s)\mathrm{d}s\,.$$

The rest of the paper is organized as follows: the main results are formulated in Section 1.2, a relation between the Kronecker and Farey sequences is discussed in Section 2, all proofs are given in Section 3.

1.2 Formulation of the Main Results

For $0 \le s,t < \infty$ define

$$\Phi_n(s,t) = \mathrm{meas}\{\theta \in [0,1) : n\delta_n(\theta) \le s,\ n\Delta_n(\theta) \le t\}\,.$$

One can interpret $\Phi_n(\cdot,\cdot)$ as the two–dimensional cumulative distribution function (c.d.f.) of the random variables $n\delta_n(\theta)$ and $n\Delta_n(\theta)$, assuming that θ is uniformly distributed on $[0,1)$.

Theorem 1 *The sequence of functions $\Phi_n(\cdot,\cdot)$ point-wisely converges, when $n \to \infty$, to the c.d.f. $\Phi(\cdot,\cdot)$ with the density*

$$\phi(s,t)=\frac{\mathrm{d}^2\Phi(s,t)}{\mathrm{d}s\mathrm{d}t}=\frac{6}{\pi^2 st}\begin{cases} s+t-1 & \text{for } 0\le s,t\le 1, s+t\ge 1\\ s(1-s)/(t-s) & \text{for } 0\le s\le 1\le t\\ t(1-t)/(s-t) & \text{for } 0\le t\le 1\le s\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

This means that for all A, measurable sets in $\mathbf{R}^2$,

$$\lim_{n\to\infty} \text{meas}\{\theta \in [0,1) : (n\delta_n(\theta), n\Delta_n(\theta)) \in A\} = \int_A \phi(s,t)\mathrm{d}s\mathrm{d}t\,.$$

Corollary 1 *The sequences of functions $n\delta_n(\theta)$ and $n\Delta_n(\theta)$ converge in distribution, when $n \to \infty$, to the probability measure with the density*

$$\phi_\beta(t) = \frac{6}{\pi^2}\begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } 0 \le t < 1 \\ \frac{t-1}{t}\log\frac{t-1}{t} + \frac{1}{t} & \text{for } t \ge 1\,. \end{cases} \tag{3}$$

Proof of Corollary 1 consists in computation of the integral $\int_0^\infty \phi(s,t)\mathrm{d}s$ where $\phi(\cdot,\cdot)$ is defined in (2).

Corollary 2 *The sequence of functions $n\alpha_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density*

$$\phi_\alpha(t) = \frac{12}{\pi^2}\begin{cases} 1 & \text{for } 0 \le t < \frac{1}{2} \\ \frac{1-t}{t}\left(1 - \log\frac{1-t}{t}\right) & \text{for } \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(Note again that the result of Corollary 2 has been proved in [2], by different arguments.)

Corollary 3 *The sequence of functions $n\beta_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density*

$$\phi_\beta(t) = \frac{12}{\pi^2}\begin{cases} 0 & \text{for } t < \frac{1}{2} \\ \frac{1-t}{t}\log\frac{1-t}{t} - \frac{1}{t} + 2 & \text{for } \frac{1}{2} \le t < 1 \\ \frac{t-1}{t}\log\frac{t-1}{t} + \frac{1}{t} & \text{for } t \ge 1\,. \end{cases} \tag{5}$$

Corollary 4 *The sequences of functions $n\gamma_n(\theta)$ and $nA_n(\theta)$ converge in distribution, when $n \to \infty$, to the probability measure with the density*

$$\phi_\gamma(t) = \frac{12}{\pi^2}\begin{cases} 0 & \text{for } t < 1 \\ \frac{t-2}{2t}\log|t-2| - \frac{t-1}{t}\log(t-1) + \frac{1}{2}\log t & \text{for } t \ge 1\,. \end{cases} \tag{6}$$

To make the difference between the asymptotic behaviour of δ_n, β_n and γ_n transparent, we provide Figure 1 which depicts the densities ϕ_δ, ϕ_β and ϕ_γ.

ϕ_β

ϕ_δ

ϕ_γ

Figure 1: Asymptotic densities for $n\delta_n(\theta)$, $n\beta_n(\theta)$ and $n\gamma_n(\theta)$.

Theorem 2 *The sequence of functions $\xi_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density*

$$\phi_\xi(t) = -\frac{12}{\pi^2}\left(\frac{\log t}{1-t} + \frac{\log(1+t)}{t}\right), \quad t \in [0,1)\,. \tag{7}$$

Theorem 2 can certainly be deduced from Theorem 1. This however would require evaluation of an unpleasant integral, in Section 3 we instead give a straightforward proof.

2 Relation with Farey Sequences and Continued Fractions

2.1 Relation with the Farey Sequences

The Farey sequence of order n, denoted by $\mathcal{F}_n$, is the collection of all rationals p/q with $p \le q$, $\gcd(p,q) = 1$ and the denominators $1 \le q \le n$. The numbers in $\mathcal{F}_n$ are arranged in the order of increase, 0 and 1 are included in $\mathcal{F}_n$ as 0/1 and 1/1 correspondingly. There are $|\mathcal{F}_n| = N(n) + 1$ points in $\mathcal{F}_n$ where

$$N(n) = \sum_{q=1}^{n} \varphi(q) = \frac{3}{\pi^2} n^2 + O(n \log(n))\,, \quad n \to \infty\,. \tag{8}$$

The following well–known statement establishes an important relation between the Kronecker and Farey sequences.

Lemma 1 (e.g. [3]) *Let* θ *be an irrational number in* $[0,1)$ *and* $W_n(\theta)$ *be the Kronecker sequence of order* n *. Let* $\{q\theta\}$ *and* $\{q'\theta\}$ *correspond respectively to the smallest and largest members of* $W_n(\theta)$*:*

$$y_1 = \delta_n(\theta) = \{q\theta\}\,, \quad y_n = 1 - \Delta_n(\theta) = \{q'\theta\}\,.$$

Define $p = \lfloor q\theta \rfloor$ *and* $p' = 1 + \lfloor q'\theta \rfloor$ *. Then* p/q *and* p'/q' *are the consecutive fractions in the Farey sequence* $\mathcal{F}_n$ *such that* $p/q < \theta < p'/q'$ *.*

Let us rewrite the quantities (1) in terms of the Farey fractions p/q and p'/q' introduced in Lemma 1:

$$\delta_n(\theta) = \{q\theta\} = q\theta - \lfloor q\theta \rfloor = q\theta - p\,, \tag{9}$$

$$\Delta_n(\theta) = 1 - \{q'\theta\} = 1 + \lfloor q'\theta \rfloor - q'\theta = p' - q'\theta\,. \tag{10}$$

This particularly implies

$$\alpha_n(\theta) = \min_{p/q \in \mathcal{F}_n} |q\theta - p|\,. \tag{11}$$

2.2 An Asymptotic Property of the Farey Sequences

In the sequel we shall use an asymptotic property of the Farey sequences formulated as Lemma 2.

If p/q and p'/q' are two consequtive Farey fractions in $\mathcal{F}_n$ then we call (q, q') a neighbouring pair of denominators. It is easy to verify that for a fixed n the set of all neighbouring pairs of denominators is

$$Q_n = \{(q, q') : q, q' \in \{1, 2, \ldots, n\}\,, \ \ \gcd(q, q') = 1\,, \ \ q + q' > n\}$$

and these pairs, properly normalised, share the asymptotic two-dimensional uniformity. Specifically, the following result holds.

Lemma 2 [1] *Let* ν_n *be the two-variate probability measure, assigning equal masses* $1/N(n)$ *to pairs* $(q/n, q'/n)$*, where* (q, q') *take all possible values in* Q_n *. Then the sequence of probability measures* $\{\nu_n\}$ *weakly converges, when* $n \to \infty$*, to the uniform probability measure on the triangle* $T = \{(x, y) : 0 \le x, y \le 1, x + y \ge 1\}$*, that is, for any continuous function* f *on* $\mathbf{R}^2$

$$\frac{1}{N(n)} \sum_{(q,q') \in Q_n} f(q/n, q'/n) \to 2 \iint_T f(x, y)\mathrm{d}x\mathrm{d}y\,, \ \ n \to \infty\,.$$

2.3 Associations with Continued Fractions

Let us now indicate an interesting analogy between the quantity $\xi_n(\theta)$ and the residuals in the continued fraction expansions.

Let θ be an irrational number in $[0,1)$. We denote by $\theta = [a_1, a_2, \ldots]$ its continued fraction expansion and by $p_n/q_n = [a_1, a_2, \ldots, a_n]$ its n-th convergent.

Let also

$$r_0 = \theta, \quad r_n = \{1/r_{n-1}\} \text{ for } n = 1, 2, \ldots$$

be the associated dynamic system. As it is well-known, the asymptotic density of $\{r_n\}$ is

$$p(t) = \frac{1}{\log 2} \frac{1}{1+t}.$$

For every $n \geq 0$, $r_n = r_n(\theta)$ allows the following continued fraction expansion: $r_n(\theta) = [a_{n+1}, a_{n+2}, \ldots]$. It is not difficult to check, see e.g. [4], that

$$r_n(\theta) = \frac{|q_n\theta - p_n|}{|q_{n-1}\theta - p_{n-1}|}, \quad n > 1.$$

This can be rewritten as

$$r_n(\theta) = \frac{\min(|q\theta - p|, |q'\theta - p'|)}{\max(|q\theta - p|, |q'\theta - p'|)},$$

where $[p/q, p'/q')$ is the current interval which θ belongs to, that is, either $[p_n/q_n, p_{n-1}/q_{n-1})$ or $[p_{n-1}/q_{n-1}, p_n/q_n)$. From this point of view, the role of $r_n(\theta)$ for $\mathcal{F}_n$ is played by

$$\xi_n(\theta) = \frac{\min(|q\theta - p|, |q'\theta - p'|)}{\max(|q\theta - p|, |q'\theta - p'|)} = \frac{\alpha_n(\theta)}{\beta_n(\theta)},$$

where p/q, p'/q' are neighbouring to θ members of $\mathcal{F}_n$. Figure 2 compares the asymptotic densities for $r_n(\theta)$ and $\xi_n(\theta)$.

3 Proofs

3.1 Proof of Theorem 1

Consider the two-variate function

$$\widetilde{\Phi}_n(s,t) = \text{meas}\{\theta \in [0,1) : n\delta_n(\theta) > s, \ n\Delta_n(\theta) > t\},$$

where $0 \leq s, t < \infty$. The c.d.f. $\Phi(s,t)$ is related to $\widetilde{\Phi}(s,t)$ throught the inclusion–exclusion formula

$$\Phi(s,t) = 1 - \widetilde{\Phi}(s,0) - \widetilde{\Phi}(0,t) + \widetilde{\Phi}(s,t). \tag{12}$$

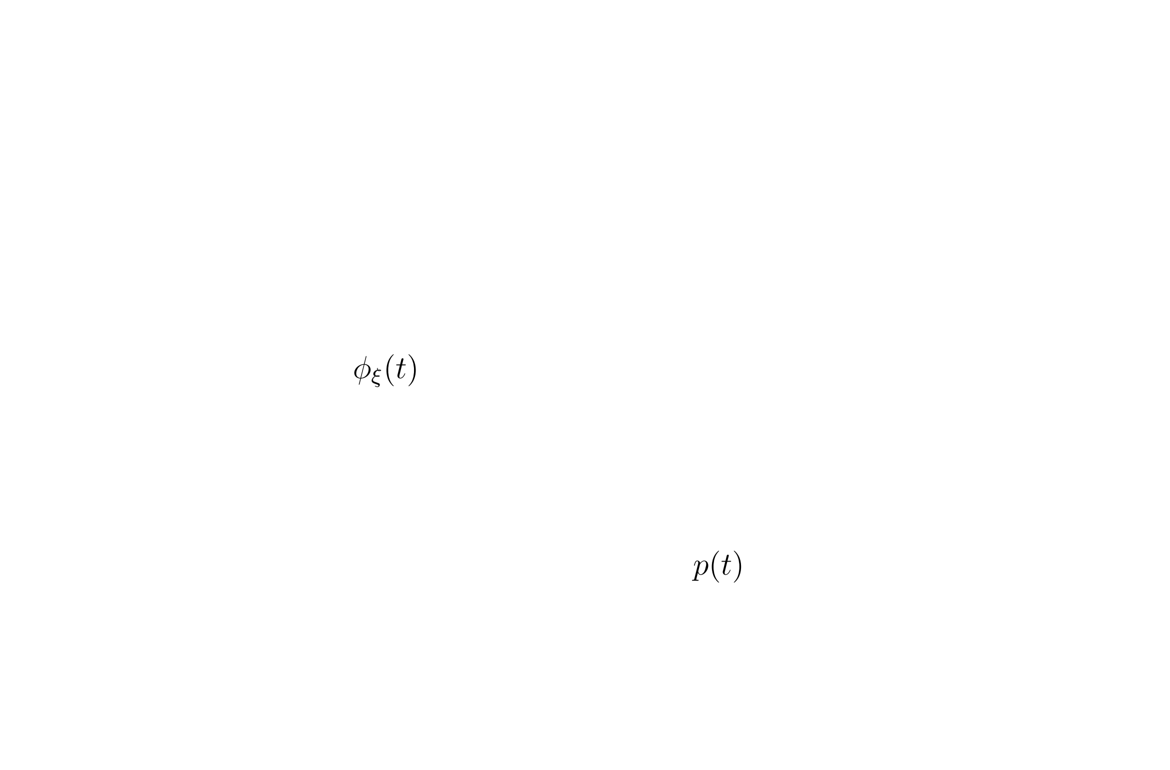

Figure 2: Asymptotic densities for sequences $r_n(\theta)$ and $\xi_n(\theta)$.

Let p/q and p'/q' be the consecutive fractions in $\mathcal{F}_n$. Define points θ_1, θ_2 in $[p/q, p'/q']$ such that, see Fig. 3,

$$n\delta_n(\theta_1) = s\,, \quad n\Delta_n(\theta_2) = t\,.$$

It is easily seen from Fig. 3 that

$$\text{meas}\,\{\theta \in [p/q, p'/q'] : n\delta_n(\theta) > s\,,\ n\Delta_n(\theta) > t\} = \begin{cases} \theta_2 - \theta_1 & \text{for} \quad \theta_2 - \theta_1 > 0 \\ 0 & \text{for} \quad \theta_2 - \theta_1 \le 0\,. \end{cases}$$

We shall now try to find a simple expression for the difference $\theta_2 - \theta_1$. First, formulas (9) and (10) yield

$$\theta_1 = \frac{s/n + p}{q}\,, \quad \theta_2 = \frac{p' - t/n}{q'}\,,$$

and therefore

$$\theta_2 - \theta_1 = \frac{p' - t/n}{q'} - \frac{s/n + p}{q} = \frac{1}{qq'}\left(1 - \frac{tq}{n} - \frac{sq'}{n}\right)\,.$$

We thus get

$$\widetilde{\Phi}_n(s,t) = \sum_{(q,q') \in Q(n,s,t)} \frac{1}{qq'}\left(1 - \frac{tq}{n} - \frac{sq'}{n}\right)\,,$$

where

$$Q(n,s,t) = \{(q,q') \in Q_n : 1 - \frac{tq}{n} - \frac{sq'}{n} > 0\}\,.$$

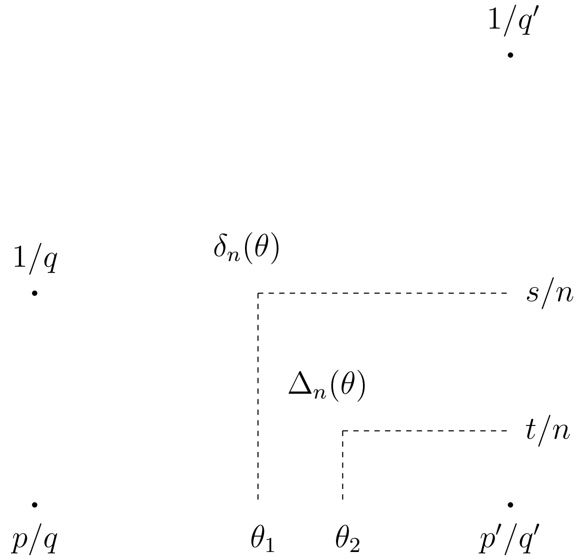

Figure 3: Functions $\delta_n(\theta)$ and $\Delta_n(\theta)$ in the interval $[p/q, p'/q']$.

Using formula (8) we have

$$\widetilde{\Phi}_n(s,t) = \frac{3}{\pi^2 N(n)} \sum_{(q,q')\in Q(n,s,t)} \frac{n^2}{qq'} \left(1 - \frac{tq}{n} - \frac{sq'}{n}\right) + O(n^{-1}\log n)\,, \quad n \to \infty\,.$$

Appling Lemma 2 we get

$$\widetilde{\Phi}_n(s,t) \to \widetilde{\Phi}(s,t) = \frac{6}{\pi^2} \iint_{Q(s,t)} \left(\frac{1 - tx - sy}{xy}\right) \mathrm{d}x\mathrm{d}y\,, \tag{13}$$

where

$$Q(s,t) = \{x, y :\ 0 \le x, y \le 1\,,\ \ x + y \ge 1\,,\ \ 1 - tx - sy > 0\}\,.$$

The formula for the integral in the right-hand side of (13) can be rewritten differently in 5 different regions:

1. For $s + t \le 1$:

$$\widetilde{\Phi}(s,t) = \frac{6}{\pi^2} \int_0^1 \int_{1-y}^1 \left(\frac{1 - tx - sy}{xy}\right) \mathrm{d}x\mathrm{d}y = 1 - \frac{6}{\pi^2}(s+t)\,.$$

2. For $0 \le s, t \le 1$, $s + t > 1$:

$$\widetilde{\Phi}(s,t) = \frac{6}{\pi^2} \int_0^{(1-s)/t} \int_{1-y}^1 \left(\frac{1 - tx - sy}{xy}\right) \mathrm{d}x\mathrm{d}y$$

$$+\frac{6}{\pi^2}\int_{(1-s)/t}^{1}\int_{1-y}^{(1-yt)/s}\left(\frac{1-tx-sy}{xy}\right)\mathrm{d}x\mathrm{d}y$$

$$=-\frac{12}{\pi^2}+\frac{6}{\pi^2}\Big(s+t+(1+\log s-s)\log\frac{1-s}{t}+(1+\log t-t)\log\frac{1-t}{s}$$

$$+\log s\log t+\mathrm{dilog\ s}+\mathrm{dilog\ t}\Big)\,.$$

3. For $s>1$, $t\le 1$:

$$\widetilde{\Phi}(s,t)=\frac{6}{\pi^2}\int_{(s-1)/(s-t)}^{1}\int_{1-y}^{(1-yt)/s}\left(\frac{1-tx-sy}{xy}\right)\mathrm{d}x\mathrm{d}y$$

$$=1-\frac{6}{\pi^2}+\frac{6}{\pi^2}\Big(t+(s-\log s-t)\log\frac{s-t}{s-1}+(1-t)\log\frac{s-1}{s}$$

$$-\mathrm{dilog\ (1-t)}+\mathrm{dilog\ }\frac{\mathrm{s(1-t)}}{\mathrm{s-t}}-\mathrm{dilog\ }\frac{\mathrm{1-t}}{\mathrm{s-t}}\Big)\,.$$

4. For $s\le 1$, $t>1$: Analogously to the previous case with the replacement $s\leftrightarrow t$.
5. For $s>1$, $t>1$: $\widetilde{\Phi}(s,t)=0$.

Using formula (12) we can find the density

$$\phi(s,t)=\frac{\mathrm{d}\Phi(s,t)}{\mathrm{d}s\mathrm{d}t}=\frac{\mathrm{d}\widetilde{\Phi}(s,t)}{\mathrm{d}s\mathrm{d}t}\,.$$

of the joint asymptotic distribution. Calculation then gives (2). ■

3.2 Proof of Corollary 2

The function $\alpha_n(\theta) = \min\{\delta_n(\theta), \Delta_n(\theta)\}$ is measurable with respect to $\mathcal{B}$, the σ-algebra of the Borel subsets of $[0,1)$, and it can be associated with the probability measure $\mathrm{d}\Phi_n^\alpha(t)$, $0 \le t < \infty$, where

$$\Phi_n^\alpha(t) = \text{meas}\{\theta \in [0,1) : n\alpha_n(\theta) \le t\} = 1 - \text{meas}\{\theta \in [0,1) : n\min(\delta_n(\theta), \Delta_n(\theta)) > t\}$$

$$= 1 - \text{meas}\{\theta \in [0,1) : n\delta_n(\theta) > t\ ,\ n\Delta_n(\theta) > t\}\,.$$

Therefore, for all $0 \le t < \infty$

$$\Phi_n^\alpha(t) \to \Phi^\alpha(t) = 1 - \widetilde{\Phi}(t,t)\,, \quad n \to \infty\,.$$

Calculation gives

$$\Phi^\alpha(t) = \begin{cases} \frac{12}{\pi^2} t & \text{for } 0 \le t < \frac{1}{2} \\ \frac{12}{\pi^2}(-t + \log\frac{1-t}{t}(t - \log t - 1) + \text{dilog}\,\frac{1}{t}) + \frac{12}{\pi^2} + 1 & \text{for } \frac{1}{2} \le t < 1 \\ 1 & \text{for } t \ge 1\,. \end{cases}$$

Differentiation gives the expression (4) for the density $\phi_\alpha(t) = \mathrm{d}\Phi^\alpha(t)/\mathrm{d}t$. ■

3.3 Proof of Corollary 3

The function $\beta_n(\theta) = \max\{\delta_n(\theta), \Delta_n(\theta)\}$ is $\mathcal{B}$-measurable. We then have for all $0 \le t < \infty$

$$\Phi_n^\beta(t) = \text{meas}\{\theta \in [0,1) : n\beta_n(\theta) \le t\} = \text{meas}\{\theta \in [0,1) : n\delta_n(\theta) \le t\ ,\ n\Delta_n(\theta) \le t\}\,.$$

Therefore, for all $0 \le t < \infty$

$$\Phi_n^\beta(t) \to \Phi^\beta(t) = \Phi(t,t)\,, \quad n \to \infty\,.$$

Calculation gives

$$\Phi^\beta(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2} \\ \frac{12}{\pi^2}(2t - \log\frac{1-t}{t}(t - \log t - 1) - \text{dilog}\,\frac{1}{t}) - \frac{12}{\pi^2} - 1 & \text{for } \frac{1}{2} \le t < 1 \\ \frac{12}{\pi^2}(\log\frac{t-1}{t}(t - \log t - 1) + \text{dilog}\,\frac{1}{t}) + \frac{12}{\pi^2} - 1 & \text{for } t \ge 1\,. \end{cases}$$

Differentiation gives the expression (5) for the density $\phi_\beta(t) = \mathrm{d}\Phi^\beta(t)/\mathrm{d}t$. ■

3.4 Proof of Corollary 4

Analogously to the proofs of Corollaries 2 and 3, the sequence of c.d.f

$$\Phi_n^\gamma(t) = \mathrm{meas}\{\theta \in [0,1) : n\gamma_n(\theta) \le t\}\,, \quad 0 \le t < \infty$$

point-wisely converges to the c.d.f

$$\Phi^\gamma(t) = \iint_{S(t)} \phi(x,y)\mathrm{d}x\mathrm{d}y\,, \quad 0 \le t < \infty\,,$$

where

$$S(t) = \{(x,y) : 0 \le x, y \le 1\,, \quad 1 \le x+y \le t\}\,.$$

Calculation yields (6).

The convergence of the sequence $\{nA_n(\theta)\}$ to the asymptotic distribution with the density ϕ_γ follows from the just proved convergence of the sequence $\{n\gamma_n(\theta)\}$ to the same distribution and the fact that $A_n(\theta) = \gamma_{n+1}(\theta)$ for all $\theta \in (0,1)$ and all $n \ge n(\theta) = \max\{1/\theta, 1/(1-\theta)\}$. ■

3.5 Proof of Theorem 2

Function $\xi_n(\theta)$ is $\mathcal{B}$-measurable. Define

$$\Phi_n^\xi(t) = \mathrm{meas}\{\theta \in [0,1) : \xi_n(\theta) \le t\}\,, \quad 0 \le t \le 1\,.$$

Let p/q, p'/q' be consecutive fractions in $\mathcal{F}_n$. Consider the behaviour of $\xi_n(\theta)$ in the interval $[p/q, p'/q']$. Define the mediant $m = (p+p')/(q+q')$. Then for θ in $[p/q, m)$ we have $\delta_n(\theta) < \Delta_n(\theta)$, for θ in $(m, p/q]$ we have $\delta_n(\theta) > \Delta_n(\theta)$ and $\delta_n(m) = \Delta_n(m)$, that is, $\xi_n(m) = 1$.

If $t \in [0,1]$ is fixed then there is a unique point θ in $[p/q, m]$ such that

$$\xi_n(\theta) = \frac{\delta_n(\theta)}{\Delta_n(\theta)} = t\,. \tag{14}$$

Easy observation shows (see Fig. 4) that

$$\mathrm{meas}\,\{\theta \in [p/q, m] : \xi_n(\theta) \le t\} = \theta - \frac{p}{q}\,.$$

Formula (14) implies

$$q\theta - p = t(p' - q'\theta)\,, \quad \theta = \frac{p+tp'}{q+tq'}\,.$$

Therefore,

$$\mathrm{meas}\,\{\theta \in [p/q, m] : \xi_n(\theta) \le t\} = \frac{p+tp'}{q+tq'} - \frac{p}{q} = \frac{t}{q(q+tq')}\,.$$

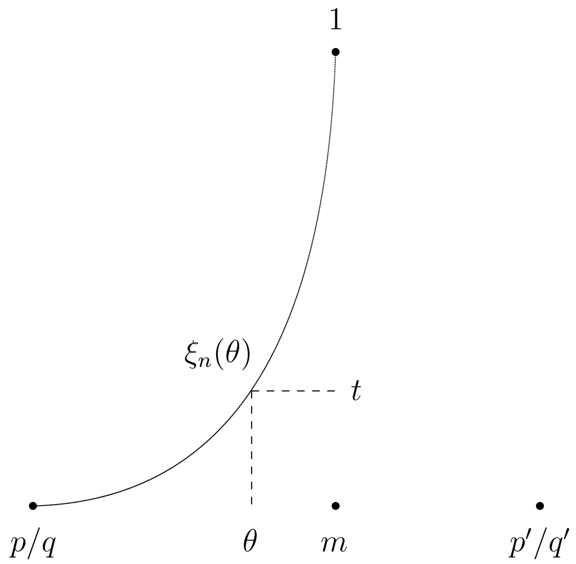

Figure 4: Behavior of function $\xi_n(\theta)$ in the interval $[p/q, p'/q']$.

We then get for all $0 \le t \le 1$

$$\mathrm{meas}\{\theta \in [0,1) : \xi_n(\theta) \le t\} = 2 \sum_{(q,q')\in Q_n} \frac{t}{q(q+tq')} = 2 \sum_{(q,q')\in Q_n} \int_0^t \frac{\mathrm{d}\tau}{(q+\tau q')^2}$$

$$= 2\int_0^t \left(\sum_{(q,q')\in Q_n} \frac{1}{(q+\tau q')^2} \right) \mathrm{d}\tau \,,$$

where factor 2 is due to the cases when $\delta_n(\theta) > \Delta_n(\theta)$.

Therefore, we can write for all $0 \le t \le 1$

$$\Phi_n^\xi(t) = \int_0^t \phi_n^\xi(\tau)\mathrm{d}\tau \,,$$

where

$$\phi_n^\xi(\tau) = 2 \sum_{(q,q')\in Q_n} \frac{1}{(q+\tau q')^2} \,.$$

Using formula (8) write

$$\phi_n^\xi(\tau) = \frac{6}{\pi^2 N(n)} \sum_{(q,q')\in Q_n} \frac{n^2}{(q+\tau q')^2} + O(n^{-1}\log n) \,, \quad n \to \infty \,.$$

Appling Lemma 2 we get that

$$\phi_n^{\xi}(\tau) \to \phi_{\xi}(\tau) = \frac{12}{\pi^2} \iint_{\substack{0 \le x,y \le 1 \\ x+y>1}} \frac{1}{(x+\tau y)^2} \mathrm{d}x\mathrm{d}y\,, \quad n \to \infty\,.$$

Calculation of the integral gives the expression (7) for the density. ■

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