Kronecker Sequences: Asymptotic Distributions of the Partition Lengths

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1 Introduction: Statement of the Problem and Formulation of the Main Results

1.1 Kronecker sequences

Let \( \theta \) be an irrational number in \([0,1)\), \( x_k = k\theta \pmod{1} \) for \( k = 1, 2, \ldots \) and let \( W_n(\theta) = \{x_1, \ldots, x_n\} \) be the Kronecker sequence of order \( n \).

In the present work we derive asymptotic distributions of different characteristics associated with the interval lengths of the partitions of \([0,1]\) generated by \( W_n(\theta) \). The main result establishes the two-dimensional asymptotic distribution of

\[
(n \min\{x_1, \ldots, x_n\}, n(1 - \max\{x_1, \ldots, x_n\}))
\]

when \( n \to \infty \). It then yields a number of results concerning the asymptotic distributions of one-dimensional characteristics.

Assume that \( y_{0,n} = 0, y_{n+1,n} = 1 \) and \( y_{k,n} \ (k = 1, \ldots, n) \) be the members of \( W_n(\theta) \) arranged in the order of increase. Define

\[
\delta_n(\theta) = y_{1,n} = \min_{k=1,\ldots,n} x_k, \quad \Delta_n(\theta) = 1 - y_{n,n} = 1 - \max_{k=1,\ldots,n} x_k
\]

and consider the partition of \([0,1]\) generated by \( W_n(\theta) \):

\[
\mathcal{P}_n(\theta) = \bigcup_{k=0}^n I_{k,n}, \quad \text{where} \ I_{k,n} = [y_{k,n}, y_{k+1,n}).
\]

It is a well known property of the Kronecker sequence, see e.g. [3],[4], that for any \( n \geq 1 \) the partition \( \mathcal{P}_n(\theta) \) of \([0,1]\) contains the intervals \( I_{k,n} \) which lengths \( |I_{k,n}| \) can only get two or three different values, namely, \( \delta_n(\theta), \Delta_n(\theta) \) and perhaps \( \delta_n(\theta) + \Delta_n(\theta) \).

Denote

\[
\alpha_n(\theta) = \min_{k=1,\ldots,n} |I_{k,n}| = \min \{\delta_n(\theta), \Delta_n(\theta)\}, \quad A_n(\theta) = \max_{k=1,\ldots,n} |I_{k,n}|,
\]

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\[ \beta_n(\theta) = \max\{\delta_n(\theta), \Delta_n(\theta)\}, \quad \gamma_n(\theta) = \delta_n(\theta) + \Delta_n(\theta), \quad \xi_n(\theta) = \frac{\alpha_n(\theta)}{\beta_n(\theta)}. \]

All these quantities, namely \( \delta_n(\theta), \Delta_n(\theta), \alpha_n(\theta), \beta_n(\theta), \gamma_n(\theta) \) and \( \xi_n(\theta) \), give a rather complete description of the partition \( \mathcal{P}_n(\theta) \). We are interested in their asymptotic behaviour, when \( n \to \infty \). The main result of the paper is formulated in Theorem 1 below and presents the joint asymptotic distribution for \( (n\delta_n(\theta), n\Delta_n(\theta)) \). In Corollaries 1–4 and Theorem 2 we derive the one-dimensional asymptotic distributions for all characteristics introduced above.

As demonstrated in Section 2, there is a close relation between the Kronecker and Farey sequences, and the quantities introduced above also characterize certain properties of the Farey sequences. (For example, \( \alpha_n(\theta) \), which asymptotic distribution has been derived in [2], characterizes the error in approximation of \( \theta \) by the Farey sequence of order \( n \), see (11).) The present paper thus also studies some distributional properties of the Farey sequences.

In what follows 'meas' stands for the Lebesgue measure on \([0,1)\), \{\cdot\} and \(\lfloor\cdot\rfloor\) denote the fractional and integer part operations correspondingly, \( \varphi(\cdot) \) is the Euler function and \( \text{dilog}(\cdot) \) is the dilogarithm function: \( \text{dilog}(t) = \int_1^t \log s/(1-s)ds \). Also, we shall say that a sequence of functions \( \psi_n(\theta), \theta \in [0,1) \), converges in distribution, when \( n \to \infty \), to a probability measure with a density \( q(\cdot) \) if for any \( t > 0 \)

\[
\lim_{n \to \infty} \text{meas}\{\theta \in [0,1) : \psi_n(\theta) \leq t\} = \int_0^t q(s)ds.
\]

The rest of the paper is organized as follows: the main results are formulated in Section 1.2, a relation between the Kronecker and Farey sequences is discussed in Section 2, all proofs are given in Section 3.

### 1.2 Formulation of the Main Results

For \( 0 \leq s, t < \infty \) define

\[
\Phi_n(s,t) = \text{meas}\{\theta \in [0,1) : n\delta_n(\theta) \leq s, n\Delta_n(\theta) \leq t\}.
\]

One can interpret \( \Phi_n(\cdot,\cdot) \) as the two–dimensional cumulative distribution function (c.d.f.) of the random variables \( n\delta_n(\theta) \) and \( n\Delta_n(\theta) \), assuming that \( \theta \) is uniformly distributed on \([0,1)\).

**Theorem 1** The sequence of functions \( \Phi_n(\cdot,\cdot) \) point-wisely converges, when \( n \to \infty \), to the c.d.f. \( \Phi(\cdot,\cdot) \) with the density

\[
\phi(s, t) = \frac{d^2 \Phi(s,t)}{ds dt} = \frac{6}{\pi^2 st} \begin{cases} 
\frac{s + t - 1}{s(1 - s)/(t - s)} & \text{for } 0 \leq s, t \leq 1, s + t \geq 1 \\
\frac{s(1 - s)/(t - s)}{t(1 - t)/(s - t)} & \text{for } 0 \leq s \leq 1 \leq t \\
0 & \text{for } 0 \leq t \leq 1 \leq s \\
\end{cases}
\]

otherwise. (2)
This means that for all $A$, measurable sets in $\mathbb{R}^2$, \[
\lim_{n \to \infty} \text{meas}\{\theta \in [0, 1) : (n\delta_n(\theta), n\Delta_n(\theta)) \in A\} = \int_A \phi(s, t) ds dt .
\]

**Corollary 1** The sequences of functions $n\delta_n(\theta)$ and $n\Delta_n(\theta)$ converge in distribution, when $n \to \infty$, to the probability measure with the density \[
\phi_\beta(t) = \frac{6}{\pi^2} \begin{cases} 
0 & \text{for } t < 0 \\
1 & \text{for } 0 \leq t < 1 \\
\frac{\log t - 1}{t} + \frac{1}{t} & \text{for } t \geq 1.
\end{cases}
\] (3)

Proof of Corollary 1 consists in computation of the integral $\int_0^\infty \phi(s, t) ds$ where $\phi(\cdot, \cdot)$ is defined in (2).

**Corollary 2** The sequence of functions $n\alpha_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density \[
\phi_\alpha(t) = \frac{12}{\pi^2} \begin{cases} 
1 & \text{for } 0 \leq t < \frac{1}{2} \\
\frac{1}{t} \left(1 - \log \frac{1-t}{t}\right) & \text{for } \frac{1}{2} \leq t < 1 \\
0 & \text{otherwise}.
\end{cases}
\] (4)

(Note again that the result of Corollary 2 has been proved in [2], by different arguments.)

**Corollary 3** The sequence of functions $n\beta_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density \[
\phi_\gamma(t) = \frac{12}{\pi^2} \begin{cases} 
0 & \text{for } t < \frac{1}{2} \\
\frac{1}{t} \log \frac{1-t}{t} - \frac{1}{2} \frac{1}{t} + 2 & \text{for } \frac{1}{2} \leq t < 1 \\
\frac{1}{t} \log \frac{1-t}{t} + \frac{1}{t} & \text{for } t \geq 1.
\end{cases}
\] (5)

**Corollary 4** The sequences of functions $n\gamma_n(\theta)$ and $n\delta_n(\theta)$ converge in distribution, when $n \to \infty$, to the probability measure with the density \[
\phi_\delta(t) = \frac{12}{\pi^2} \begin{cases} 
0 & \text{for } t < 1 \\
\frac{t^2 - 2}{2t} \log |t - 2| - \frac{1}{t} \log(t - 1) + \frac{1}{2} \log t & \text{for } t \geq 1.
\end{cases}
\] (6)

To make the difference between the asymptotic behaviour of $\delta_n$, $\beta_n$ and $\gamma_n$ transparent, we provide Figure 1 which depicts the densities $\phi_\delta$, $\phi_\beta$ and $\phi_\gamma$. 

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Theorem 2 The sequence of functions $\xi_n(\theta)$ converges in distribution, when $n \to \infty$, to the probability measure with the density

$$
\phi_{\xi}(t) = -\frac{12}{\pi^2} \left( \frac{\log t}{1-t} + \frac{\log(1+t)}{t} \right), \quad t \in [0,1).
$$

(7)

Theorem 2 can certainly be deduced from Theorem 1. This however would require evaluation of an unpleasant integral, in Section 3 we instead give a straightforward proof.

2 Relation with Farey Sequences and Continued Fractions

2.1 Relation with the Farey Sequences

The Farey sequence of order $n$, denoted by $F_n$, is the collection of all rationals $p/q$ with $p \leq q$, $\gcd(p,q) = 1$ and the denominators $1 \leq q \leq n$. The numbers in $F_n$ are arranged in the order of increase, 0 and 1 are included in $F_n$ as $0/1$ and $1/1$ correspondingly. There are $|F_n| = N(n) + 1$ points in $F_n$ where

$$
N(n) = \sum_{q=1}^{n} \varphi(q) = \frac{3}{\pi^2} n^2 + O(n \log(n)), \quad n \to \infty.
$$

(8)

The following well-known statement establishes an important relation between the Kronecker and Farey sequences.
Lemma 1 (e.g. [3]) Let \( \theta \) be an irrational number in \([0,1)\) and \( W_n(\theta) \) be the Kronecker sequence of order \( n \). Let \( \{q\theta\} \) and \( \{q\theta\}' \) correspond respectively to the smallest and largest members of \( W_n(\theta) \):

\[
y_1 = \delta_n(\theta) = \{q\theta\}, \quad y_n = 1 - \Delta_n(\theta) = \{q\theta\}'.
\]

Define \( p = \lfloor q\theta \rfloor \) and \( p' = 1 + \lfloor q\theta \rfloor \). Then \( p/q \) and \( p'/q' \) are the consecutive fractions in the Farey sequence \( F_n \) such that \( p/q < \theta < p'/q' \).

Let us rewrite the quantities (1) in terms of the Farey fractions \( p/q \) and \( p'/q' \) introduced in Lemma 1:

\[
\delta_n(\theta) = \{q\theta\} = q\theta - \lfloor q\theta \rfloor = q\theta - p, \quad p = \lfloor q\theta \rfloor,
\]

\[
\Delta_n(\theta) = 1 - \{q\theta\}' = 1 + \lfloor q\theta \rfloor - q\theta = p' - q\theta, \quad p' = 1 + \lfloor q\theta \rfloor.
\]

This particularly implies

\[
\alpha_n(\theta) = \min_{p/q \in F_n} |q\theta - p|.
\]

2.2 An Asymptotic Property of the Farey Sequences

In the sequel we shall use an asymptotic property of the Farey sequences formulated as Lemma 2.

If \( p/q \) and \( p'/q' \) are two consecutive Farey fractions in \( F_n \) then we call \((q,q')\) a neighbouring pair of denominators. It is easy to verify that for a fixed \( n \) the set of all neighbouring pairs of denominators is

\[
Q_n = \{(q,q') : q,q' \in \{1,2,\ldots,n\}, \gcd(q,q') = 1, \quad q + q' > n\}
\]

and these pairs, properly normalised, share the asymptotic two-dimensional uniformity. Specifically, the following result holds.

Lemma 2 [1] Let \( \nu_n \) be the two-variate probability measure, assigning equal masses \( 1/N(n) \) to pairs \((q/n,q'/n)\), where \((q,q')\) take all possible values in \( Q_n \). Then the sequence of probability measures \( \{\nu_n\} \) weakly converges, when \( n \to \infty \), to the uniform probability measure on the triangle \( T = \{(x,y) : 0 \leq x, y \leq 1, x + y \geq 1\} \), that is, for any continuous function \( f \) on \( \mathbb{R}^2 \)

\[
\frac{1}{N(n)} \sum_{(q,q') \in Q_n} f(q/n,q'/n) \to 2 \int_T f(x,y)dx dy, \quad n \to \infty.
\]
2.3 Associations with Continued Fractions

Let us now indicate an interesting analogy between the quantity \( \xi_n(\theta) \) and the residuals in the continued fraction expansions.

Let \( \theta \) be an irrational number in \([0, 1)\). We denote by \( \theta = [a_1, a_2, \ldots] \) its continued fraction expansion and by \( p_n/q_n = [a_1, a_2, \ldots, a_n] \) its \( n \)-th convergent.

Let also \( r_0 = \theta, \quad r_n = \{1/r_{n-1}\} \) for \( n = 1, 2, \ldots \)

be the associated dynamic system. As it is well-known, the asymptotic density of \( \{r_n\} \) is

\[
p(t) = \frac{1}{\log 2 - 1 + t}.
\]

For every \( n \geq 0, r_n = r_n(\theta) \) allows the following continued fraction expansion:

\[
r_n(\theta) = \frac{|q_n \theta - p_n|}{|q_{n-1} \theta - p_{n-1}|}, \quad n > 1.
\]

This can be rewritten as

\[
r_n(\theta) = \text{min}(|q_\theta - p|, |q_\theta - p'|) / \text{max}(|q_\theta - p|, |q_\theta - p'|),
\]

where \([p/q, p'/q']\) is the current interval which \( \theta \) belongs to, that is, either \([p_n/q_n, p_{n-1}/q_{n-1}]\) or \([p_{n-1}/q_{n-1}, p_n/q_n]\). From this point of view, the role of \( r_n(\theta) \) for \( F_n \) is played by

\[
\xi_n(\theta) = \frac{\text{min}(|q_\theta - p|, |q_\theta - p'|)}{\text{max}(|q_\theta - p|, |q_\theta - p'|)} = \alpha_n(\theta) / \beta_n(\theta),
\]

where \( p/q, p'/q' \) are neighbouring to \( \theta \) members of \( F_n \). Figure 2 compares the asymptotic densities for \( r_n(\theta) \) and \( \xi_n(\theta) \).

3 Proofs

3.1 Proof of Theorem 1

Consider the two-variate function

\[
\tilde{\Phi}(s, t) = \text{meas}\{\theta \in [0, 1) : n\delta_n(\theta) > s, \quad n\Delta_n(\theta) > t\},
\]

where \( 0 \leq s, t < \infty \). The c.d.f. \( \Phi(s, t) \) is related to \( \tilde{\Phi}(s, t) \) through the inclusion–exclusion formula

\[
\Phi(s, t) = 1 - \tilde{\Phi}(s, 0) - \tilde{\Phi}(0, t) + \tilde{\Phi}(s, t).
\]
Let $p/q$ and $p'/q'$ be the consecutive fractions in $\mathcal{F}_n$. Define points $\theta_1, \theta_2$ in $[p/q, p'/q']$ such that, see Fig. 3,

$$n\delta_n(\theta_1) = s, \ n\Delta_n(\theta_2) = t.$$ 

It is easily seen from Fig. 3 that

$$\text{meas} \{ \theta \in [p/q, p'/q'] : n\delta_n(\theta) > s, \ n\Delta_n(\theta) > t \} = \begin{cases} \theta_2 - \theta_1 & \text{for } \theta_2 - \theta_1 > 0 \\ 0 & \text{for } \theta_2 - \theta_1 \leq 0. \end{cases}$$ 

We shall now try to find a simple expression for the difference $\theta_2 - \theta_1$. First, formulas (9) and (10) yield

$$\theta_1 = \frac{s/n + p}{q}, \ \theta_2 = \frac{p' - t/n}{q'},$$

and therefore

$$\theta_2 - \theta_1 = \frac{p' - t/n}{q'} - \frac{s/n + p}{q} = \frac{1}{qq'} \left( 1 - \frac{tq}{n} - \frac{sq'}{n} \right).$$

We thus get

$$\tilde{\Phi}_n(s, t) = \sum_{(q, q') \in Q(n, s, t)} \frac{1}{qq'} \left( 1 - \frac{tq}{n} - \frac{sq'}{n} \right),$$

where

$$Q(n, s, t) = \{(q, q') \in Q_n : 1 - \frac{tq}{n} - \frac{sq'}{n} > 0\}.$$
Using formula (8) we have
\[
\tilde{\Phi}_n(s, t) = \frac{3}{\pi^2 N(n)} \sum_{(q, q') \in Q(n, s, t)} n^2 \left( \frac{1 - \frac{tq}{n} - \frac{sq'}{n}}{qq'} \right) + O(n^{-1} \log n), \ n \to \infty.
\]

Applying Lemma 2 we get
\[
\tilde{\Phi}_n(s, t) \to \tilde{\Phi}(s, t) = \frac{6}{\pi^2} \int \int_{Q(s, t)} \left( \frac{1 - tx - sy}{xy} \right) \, dx \, dy,
\]
where
\[
Q(s, t) = \{ x, y : 0 \leq x, y \leq 1, \ x + y \geq 1, \ 1 - tx - sy > 0 \}.
\]
The formula for the integral in the right-hand side of (13) can be rewritten differently in 5 different regions:

1. For \( s + t \leq 1 \):
\[
\tilde{\Phi}(s, t) = \frac{6}{\pi^2} \int_0^1 \int_{1-y}^1 \left( \frac{1 - tx - sy}{xy} \right) \, dx \, dy = 1 - \frac{6}{\pi^2} (s + t).
\]

2. For \( 0 \leq s, t \leq 1, s + t > 1 \):
\[
\tilde{\Phi}(s, t) = \frac{6}{\pi^2} \int_0^{(1-s)/t} \int_{1-y}^1 \left( \frac{1 - tx - sy}{xy} \right) \, dx \, dy
\]
\[ + \frac{6}{\pi^2} \int_{1-s/t}^{1} \int_{1-y}^{(1-yt)/s} \left( \frac{1-tx-sy}{xy} \right) dxdy \]

\[ = -\frac{12}{\pi^2} + \frac{6}{\pi^2} \left( s + t + (1 + \log s - s) \log \frac{1-s}{t} + (1 + \log t - t) \log \frac{1-t}{s} \right. \]

\[ + \log s \log t + \text{dilog } s + \text{dilog } t \bigg). \]

3. For \( s > 1, t \leq 1 \):

\[ \tilde{\Phi}(s, t) = \frac{6}{\pi^2} \int_{(s-1)/t}^{1} \int_{1-y}^{(1-yt)/s} \left( \frac{1-tx-sy}{xy} \right) dxdy \]

\[ = 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2} \left( t + (s - \log s - t) \log \frac{s-t}{s-1} + (1-t) \log \frac{s-1}{s} \right) \]

\[ - \text{dilog } (1-t) + \text{dilog } \frac{s(1-t)}{s-t} - \text{dilog } \frac{1-t}{s-t} \bigg). \]

4. For \( s \leq 1, t > 1 \): Analogously to the previous case with the replacement \( s \leftrightarrow t \).

5. For \( s > 1, t > 1 \): \( \tilde{\Phi}(s, t) = 0 \).

Using formula (12) we can find the density

\[ \phi(s, t) = \frac{d\Phi(s, t)}{dsdt} = \frac{d\tilde{\Phi}(s, t)}{dsdt} . \]

of the joint asymptotic distribution. Calculation then gives (2).
3.2 Proof of Corollary 2

The function \( \alpha_n(\theta) = \min\{\delta_n(\theta), \Delta_n(\theta)\} \) is measurable with respect to \( \mathcal{B} \), the \( \sigma \)-algebra of the Borel subsets of \([0, 1)\), and it can be associated with the probability measure \( d\Phi_n^\alpha(t) \), \( 0 \leq t < \infty \), where

\[
\Phi_n^\alpha(t) = \text{meas}\{\theta \in [0, 1) : n\alpha_n(\theta) \leq t\} = 1 - \text{meas}\{\theta \in [0, 1) : n\min(\delta_n(\theta), \Delta_n(\theta)) > t\}
\]

\[
= 1 - \text{meas}\{\theta \in [0, 1) : n\delta_n(\theta) > t \text{, } n\Delta_n(\theta) > t\}.
\]

Therefore, for all \( 0 \leq t < \infty \)

\[
\Phi_n^\alpha(t) \to \Phi^\alpha(t) = 1 - \tilde{\Phi}(t, t) \quad n \to \infty.
\]

Calculation gives

\[
\Phi^\alpha(t) = \begin{cases} 
\frac{12}{\pi^2} t & \text{for } 0 \leq t < \frac{1}{2} \\
\frac{12}{\pi^2} \left(-t + \log\frac{1-t}{t} + \mathrm{dilog}\frac{1}{t} - 1\right) + \frac{12}{\pi^2} + 1 & \text{for } \frac{1}{2} \leq t < 1 \\
1 & \text{for } t \geq 1.
\end{cases}
\]

Differentiation gives the expression (4) for the density \( \phi^\alpha(t) = d\Phi^\alpha(t)/dt \).

3.3 Proof of Corollary 3

The function \( \beta_n(\theta) = \max\{\delta_n(\theta), \Delta_n(\theta)\} \) is \( \mathcal{B} \)-measurable. We then have for all \( 0 \leq t < \infty \)

\[
\Phi_n^\beta(t) = \text{meas}\{\theta \in [0, 1) : n\beta_n(\theta) \leq t\} = \text{meas}\{\theta \in [0, 1) : n\delta_n(\theta) \leq t \text{, } n\Delta_n(\theta) \leq t\}.
\]

Therefore, for all \( 0 \leq t < \infty \)

\[
\Phi_n^\beta(t) \to \Phi^\beta(t) = \Phi(t, t) \quad n \to \infty.
\]

Calculation gives

\[
\Phi^\beta(t) = \begin{cases} 
0 & \text{for } 0 \leq t < \frac{1}{2} \\
\frac{12}{\pi^2} \left(2\log\frac{1-t}{t} - 1 - \frac{\log\frac{1-t}{t} - 1}{t} - 1\right) + \frac{12}{\pi^2} - 1 & \text{for } \frac{1}{2} \leq t < 1 \\
\frac{12}{\pi^2} \left(\log\frac{t-1}{t} - 1 + \frac{\log\frac{t-1}{t} - 1}{t} + \frac{12}{\pi^2} - 1\right) & \text{for } t \geq 1.
\end{cases}
\]

Differentiation gives the expression (5) for the density \( \phi^\beta(t) = d\Phi^\beta(t)/dt \).
3.4 Proof of Corollary 4

Analogously to the proofs of Corollaries 2 and 3, the sequence of c.d.f
\[ \Phi_{n\gamma}(t) = \text{meas}\{\theta \in [0,1) : n\gamma(\theta) \leq t\} , \quad 0 \leq t < \infty \]
point-wisely converges to the c.d.f
\[ \Phi(\gamma(t) = \int \int_{S(t)} \phi(x,y)dxdy , \quad 0 \leq t < \infty , \]
where
\[ S(t) = \{(x,y) : 0 \leq x, y \leq 1 , \quad 1 \leq x + y \leq t\} . \]
Calculation yields (6).

The convergence of the sequence \( \{nA_n(\theta)\} \) to the asymptotic distribution with the density \( \phi\) follows from the just proved convergence of the sequence \( \{n\gamma(\theta)\} \) to the same distribution and the fact that \( A_n(\theta) = \gamma_{n+1}(\theta) \) for all \( \theta \in (0,1) \) and all \( n \geq n(\theta) = \max\{1/\theta, 1/(1-\theta)\} \).

3.5 Proof of Theorem 2

Function \( \xi_n(\theta) \) is \( B \)-measurable. Define
\[ \Phi^\xi_n(t) = \text{meas}\{\theta \in [0,1) : \xi_n(\theta) \leq t\} , \quad 0 \leq t \leq 1 . \]
Let \( p/q, p'/q' \) be consecutive fractions in \( F_n \). Consider the behaviour of \( \xi_n(\theta) \) in the interval \( [p/q, p'/q'] \). Define the mediant \( m = (p + p')/(q + q') \). Then for \( \theta \) in \( [p/q, m) \) we have \( \delta_n(\theta) < \Delta_n(\theta) \), for \( \theta \) in \( (m, p/q] \) we have \( \delta_n(\theta) > \Delta_n(\theta) \) and \( \delta_n(m) = \Delta_n(m) \), that is, \( \xi_n(m) = 1 \).

If \( t \in [0,1] \) is fixed then there is a unique point \( \theta \) in \( [p/q, m] \) such that
\[ \xi_n(\theta) = \frac{\delta_n(\theta)}{\Delta_n(\theta)} = t . \quad (14) \]
Easy observation shows (see Fig. 4) that
\[ \text{meas}\{\theta \in [p/q, m] : \xi_n(\theta) \leq t\} = \theta - \frac{p}{q} . \]
Formula (14) implies
\[ q\theta - p = t(p' - q'\theta) , \quad \theta = \frac{p + tp'}{q + tq'} . \]
Therefore,
\[ \text{meas}\{\theta \in [p/q, m] : \xi_n(\theta) \leq t\} = \frac{p + tp'}{q + tq'} - \frac{p}{q} = \frac{t}{q(q + tq')} . \]
We then get for all $0 \leq t \leq 1$

$$\text{meas}\{\theta \in [0, 1) : \xi_n(\theta) \leq t\} = 2 \sum_{(q,q') \in Q_n} \frac{t}{q(q + tq')} = 2 \sum_{(q,q') \in Q_n} \int_0^t \frac{d\tau}{(q + \tau q')^2}$$

$$= 2 \int_0^t \left( \sum_{(q,q') \in Q_n} \frac{1}{(q + \tau q')^2} \right) d\tau,$$

where factor 2 is due to the cases when $\delta_n(\theta) > \Delta_n(\theta)$.

Therefore, we can write for all $0 \leq t \leq 1$

$$\Phi_n^\xi(t) = \int_0^t \phi_n^\xi(\tau) d\tau,$$

where

$$\phi_n^\xi(\tau) = 2 \sum_{(q,q') \in Q_n} \frac{1}{(q + \tau q')^2}.$$

Using formula (8) write

$$\phi_n^\xi(\tau) = \frac{6}{\pi^2 N(n)} \sum_{(q,q') \in Q_n} \frac{n^2}{(q + \tau q')^2} + O(n^{-1} \log n), \ n \to \infty.$$
Applying Lemma 2 we get that
\[
\phi_n^\xi(\tau) \to \phi^\xi(\tau) = \frac{12}{\pi^2} \int_{x+y \leq 1} \int_{x+y > 1} \frac{1}{(x+\tau y)^2} \, dx \, dy , \quad n \to \infty .
\]

Calculation of the integral gives the expression (7) for the density. \hfill \blacksquare

\textbf{References}


