Approximation of real numbers by rationals: some metric theorems

Pavel Kargaev Anatoly Zhigljavsky

Abstract

Let x be a real number in $[0,1]$, $\mathcal{F}_n$ be the Farey sequence of order n and $\rho_n(x)$ be the distance between x and $\mathcal{F}_n$. The first result concerns the average rate of approximation:

$$\int_0^1 \rho_n(x)dx = \frac{3}{\pi^2}\frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right), \quad n \to \infty\,.$$

The second result states that any badly approximable number is better approximable by rationals than all numbers in average. Namely, we show that if $x \in [0,1]$ is a badly approximable number then $c_1 \le n^2\rho_n(x) \le c_2$ for all integers $n \ge 1$ and some constants $c_1 > 0, c_2 > 0$. The last two theorems can be considered as analogues of Khinchin's metric theorem regarding the behaviour of inferior and superior limits of $n^2\rho_n(x)f(\log n)$, when $n \to \infty$, for almost all $x \in [0,1]$ and suitable functions $f(\cdot)$.

Key words: Farey sequence, rational approximation, badly approximable numbers, diophantine approximation, Khinchin's metric theorem.

1 Introduction: Statement of the problem and formulation of the main results

Let x be a real number in $[0,1]$ and $\mathcal{F}_n$ be the Farey sequence of order n, that is, the collection of all rationals p/q with $p \le q$, $(p,q) = 1$ and denominators $q \le n$. Let

$$\rho_n(x) = \min_{\frac{p}{q} \in \mathcal{F}_n} \left| x - \frac{p}{q} \right| \tag{1}$$

be the distance between x and $\mathcal{F}_n$. For fixed n and x, $\rho_n(x)$ is the inaccuracy of approximation of a real x by the rationals with denominators bounded by n. We are interested in the asymptotic behaviour, when $n \to \infty$, of $\rho_n(x)$. Specifically, we are interested in the asymptotics of the average inaccuracy

$$E_n = \int_0^1 \rho_n(x)dx \tag{2}$$

as well as in the inferior and superior limits

$$\liminf_{n\to\infty} n^2\rho_n(x)f(\log n), \quad \limsup_{n\to\infty} n^2\rho_n(x)/f(\log n)$$

for suitable functions $f(\cdot)$, for almost all (a.a.) x with respect to the Lebesgue measure on [0,1] and for x in the class of badly approximable numbers.

The first result of the paper is

Theorem 1 *The average inaccuracy* (2) *asymptotically equals*

$$E_n = \frac{3}{\pi^2}\frac{\log n}{n^2} + O\left(\frac{1}{n^2}\right), \qquad n \to \infty\,. \tag{3}$$

The number of different terms in the Farey sequence $\mathcal{F}_n$ is

$$|\mathcal{F}_n| = \sum_{k=1}^{n} \varphi(k) + 1 = \frac{3n^2}{\pi^2} + O(n\log n), \qquad n \to \infty\,, \tag{4}$$

where $\varphi(\cdot)$ is the Euler function. (The asymptotic expression (4) is a well-known formula in number theory, see [2].) If we take $|\mathcal{F}_n|$ equidistant points in [0,1], including both endpoints, then the average inaccuracy of the resulting approximation is

$$\frac{1}{4(|\mathcal{F}_n|-1)} \sim \frac{\pi^2}{12n^2}, \qquad n \to \infty$$

which is of better order. Therefore the Farey sequences do not provide the best order of approximation of real numbers in [0,1], in average. The next natural question about precision of the approximation by rationals concerns the asymptotic behaviour of $\rho_n(x)$ for x in different classes of irrational numbers. As an example we consider the class of the so called *badly approximable numbers* which, as it is well known, has the cardinality of the continuum and contains all second order irrationals.

Theorem 2 *Assume that $x \in [0,1]$ is a badly approximable number; that is, there exists a constant $c > 0$ such that*

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2} \tag{5}$$

for all integers $q \geq 1$. Then there exist constants $c_1 > 0, c_2 > 0$ such that

$$\frac{c_1}{n^2} \leq \rho_n(x) \leq \frac{c_2}{n^2} \tag{6}$$

for all integers $n \geq 1$.

Comparison of Theorem 1 and Theorem 2 implies that in fact the badly approximable numbers are better approximable by rationals than all numbers in average.

The following statement is a simple consequence of the classical Khinchin's metric theorem.

Theorem 3 *Let $f(\cdot)$ be an increasing function on $[1,\infty)$ and $f(1)>0$. Then*
(i) *if the integral*

$$\int_1^\infty \frac{dt}{f(t)} \tag{7}$$

diverges then for a.a. $x\in[0,1]$ the inequality

$$\rho_n(x) \le \frac{1}{n^2 f(\log n)} \tag{8}$$

holds for infinitely many integers n;
(ii) *if the integral* (7) *converges then for a.a. $x\in[0,1]$ the inequality* (8) *holds only for finitely many integers n.*

In terms of the inferior limit the statement of Theorem 3 can be written as

$$\liminf_{n\to\infty} \rho_n(x) n^2 f(\log n) = \begin{cases} 0 & \text{in case (i)} \\ +\infty & \text{in case (ii)} \end{cases}$$

for a.a. $x\in[0,1]$. This particularly implies

$$\lim_{n\to\infty} \rho_n(x) n^2 \log^{1+\varepsilon} n = +\infty, \quad \liminf_{n\to\infty} \rho_n(x) n^2 \log n = 0 \tag{9}$$

for a.a. $x\in[0,1]$ and any $\varepsilon>0$.

The authors have met the biggest technical difficulties while proving the following theorem which is an analogue of Theorem 3 in the case of the superior limit of $\rho_n(x)$.

Theorem 4 *Let, analogously to the statement of* Theorem 3, *$f(\cdot)$ be an increasing function on $[1,\infty)$ and $f(1)>0$. Then*
(i) *if the integral* (7) *diverges and*

$$\limsup_{t\to\infty} \frac{\log f(t)}{t} < 1 \tag{10}$$

then for a.a. $x\in[0,1]$ the inequality

$$\rho_n(x) \ge \frac{f(\log n)}{n^2} \tag{11}$$

holds for infinitely many integers n;
(ii) *if the integral* (7) *converges then for a.a. $x\in[0,1]$ the inequality* (11) *holds only for finitely many integers n.*

In terms of the inferior limit the statement of Theorem 4 can be written as

$$\limsup_{n\to\infty} \frac{\rho_n(x) n^2}{f(\log n)} = \begin{cases} +\infty & \text{in case (i)} \\ 0 & \text{in case (ii)} \end{cases}$$

for a.a. $x \in [0,1]$. This particularly implies

$$\limsup_{n\to\infty} \frac{\rho_n(x)n^2}{\log n} = +\infty, \quad \lim_{n\to\infty} \frac{\rho_n(x)n^2}{\log^{1+\varepsilon} n} = 0 \tag{12}$$

for a.a. $x \in [0,1]$ and any $\varepsilon > 0$.

It is worthwhile to mention that the regularity condition (10) is rather weak. Note that in view of $\rho_n(x) \geq 1/n$, the inequality (11) yields $f(\log n) \leq n$, hence $\log f(t) \leq t$ with $t = \log n$.

It is interesting to note that comparison of (3), (9) and (12) leads to an observation that the asymptotic behaviour of the average inaccuracy $\int \rho_n(x)dx$ resembles the behaviour of the superior limit of $\rho_n(x)$ more than that of the inferior limit.

The main results of the present work can be regarded as metric theorems in the theory of diophantine approximations. Roughly speaking, the difference between classical results and our results is that we are interested in answering "How well are irrationals $x \in [0,1]$ approximated by the rationals with denominators bounded by some number n ?" and the classical results on diophantine approximations typically answer the questions like: "How often can x be approximated by rationals p/q with a precision bounded by a given function of q?"

2 Some properties of Farey sequences and proofs of Theorems 1,2

2.1 Farey sequences and their properties

Let n be a fixed integer. The Farey sequence $\mathcal{F}_n$ of order n is the increasing sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus p/q belongs to $\mathcal{F}_n$ if $0 \leq p \leq q \leq n$ and $(p,q) = 1$, that is, the numbers p and q have no common factors bigger than 1. The numbers 0 and 1 are included into $\mathcal{F}_n$ in the form 0/1 and 1/1. We refer to [2, 6] for proofs of the formulated below properties of Farey sequences and further discussions on their properties.

Write

$$N(n) = \sum_{k=1}^{n} \varphi(k)\,. \tag{13}$$

where $\varphi(\cdot)$ is the Euler function: $\varphi(k)$ is the number of positive integers relatively prime with k. Then the number of terms in $\mathcal{F}_n$ is $|\mathcal{F}_n| = N(n) + 1$.

The mediant of two fractions a/b and e/f is defined as $(a+e)/(b+f)$ and it always lies within the interval $(a/b, e/f)$. Each non-integer term in a Farey sequence $\{\ldots, a/b, c/d, e/f, \ldots\}$ is the mediant of its two neighbours: $c/d = (a+e)/(b+f)$.

Other important properties of Farey sequences relate two succesive terms: if a/b and c/d are two succesive terms in $\mathcal{F}_n$ then

$$bc - ad = 1,\ b + d > n,\ b \neq d\,. \tag{14}$$

In the literature on diophantine approximations it is often stated that Farey sequences provide good approximations to real numbers in [0,1] with a reference to the following two properties: for any $x \in (0,1)$

Dirichlet (1842) [1]: there exists a fraction $p/q \in \mathcal{F}_n$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q(n+1)}\,, \tag{15}$$

Hurwitz (1891) [3]: there exist infinitely many integers p, q such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}\,. \tag{16}$$

References to the property (15) are sometimes accompained with the words that "Farey sequences have certain uniformity which explains their importance". However (15) and (16) do not directly characterise uniformity of Farey sequences and the approach of the present work is an attempt to rigorously measure the "uniformity" of Farey sequences and their properties as approximation sequences.

Denote by

$$0 = x_{1,n} < x_{2,n} < \dots x_{N(n),n} < x_{N(n)+1,n} = 1$$

the elements of $\mathcal{F}_n$. We shall call the partition $\mathcal{P}_n$ of [0,1), generated by $\mathcal{F}_n$, the *Farey partition* of order n:

$$\mathcal{P}_n \ : \quad [0,1) = \bigcup_{i=1}^{N(n)} [x_{i,n}, x_{i+1,n})\,.$$

In addition to the partition $\mathcal{P}_n$, consisting of $N(n)$ subintervals, we will also need the partition of [0,1) onto $2N(n)$ intervals generated by $x_{i,n}$, the elements of $\mathcal{F}_n$, and the midpoints $\frac{1}{2}(x_{i,n}+x_{i+1,n})$ for $i = 1,\dots,N(n)$:

$$\mathcal{R}_n \ : \quad [0,1) = \bigcup_{i=1}^{N(n)} \left(\left[x_{i,n}, \frac{x_{i,n}+x_{i+1,n}}{2} \right) \bigcup \left[\frac{x_{i,n}+x_{i+1,n}}{2}, x_{i+1,n} \right) \right)$$

2.2 Proof of Theorem 1.

The lengths of the intervals $[x_{i,n}, x_{i+1,n})$ of the partition $\mathcal{P}_n$ equal $p_{i,n} = x_{i+1,n} - x_{i,n}$ for $i = 1,\dots,N(n)$ and satisfy

$$p_{i,n} > 0, \quad \sum_{i=1}^{N(n)} p_{i,n} = 1\,.$$

Rewrite the average inaccuracy (2) as follows

$$E_n = \int_0^1 \rho_n(x)dx = \int_0^1 \min_{x_{i,n}\in\mathcal{F}_n} |x - x_{i,n}|dx =$$

$$= \sum_{i=1}^{N(n)} \int_{x_{i,n}}^{\frac{1}{2}(x_{i,n}+x_{i+1,n})} (x - x_{i,n})dx + \sum_{i=1}^{N(n)} \int_{\frac{1}{2}(x_{i,n}+x_{i+1,n})}^{x_{i+1,n}} (x_{i+1,n} - x)dx =$$

$$= 2\sum_{i=1}^{N(n)} \int_{x_{i,n}}^{\frac{1}{2}(x_{i,n}+x_{i+1,n})} (x - x_{i,n})dx = 2\sum_{i=1}^{N(n)} \int_0^{\frac{1}{2}p_{i,n}} xdx = \frac{1}{4}\sum_{i=1}^{N(n)} p_{i,n}^2\,.$$

Consider the Farey partition $\mathcal{P}_n$. The property (14) of $\mathcal{F}_n$ implies that if the endpoints of $[x_{i,n}, x_{i+1,n})$, that is, $x_{i,n}$ and $x_{i+1,n}$, have denominators q and q' and $q \le q'$ then $q' > n/2$ and the length

$$p_{i,n} = x_{i+1,n} - x_{i,n} = 1/(qq')$$

of $[x_{i,n}, x_{i+1,n})$ can always be bounded as

$$\frac{1}{qn} \leq p_{i,n} \leq \frac{1}{q(n-q)} \,. \tag{17}$$

We shall use these bounds when one of the endpoints of $[x_{i,n}, x_{i+1,n})$ has a denominator $q \leq n/2$. The total number of intervals in $\mathcal{P}_n$ with this property equals

$$N_n^{'} = \sum_{q=1}^{m} 2\varphi(q) = \frac{3n^2}{2\pi^2} + O(n\log n), \quad n \to \infty\,, \tag{18}$$

where we have introduced the notation $m = \lfloor n/2 \rfloor$.

An upper bound for the length of the intervals $[x_{i,n}, x_{i+1,n})$, when both endpoints have denominators $> n/2$, follows from $p_{i,n} = 1/(qq')$:

$$p_{i,n} \leq \frac{4}{n^2} \,. \tag{19}$$

The bounds (17), (19) for $p_{i,n}$ give the following lower and upper bounds for E_n:

$$4E_n \geq A = \sum_{q=1}^{m} 2\varphi(q)\frac{1}{q^2n^2} \tag{20}$$

$$4E_n \leq B = \sum_{q=1}^{m} 2\varphi(q)\frac{1}{q^2(n-q)^2} + \frac{16}{n^4}\sum_{q=m+1}^{n} 2\varphi(q) \tag{21}$$

Applying (4) we obtain

$$B = \sum_{q=1}^{m} 2\varphi(q)\frac{1}{q^2(n-q)^2} + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

Therefore

$$0 \leq B - A \leq 2\sum_{q=1}^{m}\varphi(q)\frac{n^2-(n-q)^2}{q^2n^2(n-q)^2} + O\left(\frac{1}{n^2}\right) \leq \frac{16}{n^3}\sum_{q=1}^{m}\frac{\varphi(q)}{q} + O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^2}\right)$$

when $n \to \infty$.

Using Abel transformation, represent A in the form

$$A = \frac{2}{n^2}\sum_{q=1}^{m}\frac{\varphi(q)}{q^2} = \frac{2}{n^2}\left(\sum_{q=1}^{m-1} N(q)\left(\frac{1}{q^2} - \frac{1}{(q+1)^2}\right) + N(m)\frac{1}{m^2}\right) .$$

Again using (4) represent $N(q)$ in the form

$$N(q) = \frac{6}{\pi^2}\frac{q(q+1)}{2} + O(q(\log q + 1)), \quad q \to \infty.$$

This implies

$$A = \frac{12}{\pi n^2}\left(\sum_{q=1}^{m-1}\frac{q(q+1)}{2}\left(\frac{1}{q^2} - \frac{1}{(q+1)^2}\right) + \frac{m(m+1)}{2}\frac{1}{m^2}\right) + O\left(\frac{1}{n^2}\right) =$$

$$= \frac{12}{\pi n^2}\sum_{q=1}^{m}\frac{1}{q} + O\left(\frac{1}{n^2}\right) = \frac{12\log n}{\pi n^2} + O\left(\frac{1}{n^2}\right), \quad n \to \infty.$$

This yields (3). $\square$

2.3 Proof of Theorem 2

Proof. Let $x \in [0,1]$ be badly approximable and let $p_1/q_1, p_2/q_2, \ldots$ be the convergents of its continued fraction expansion. Since x is badly approximable, $q_{k+1} \leq Kq_k$ for $k = 1, 2, \ldots$ with $K \leq 1/c$, where c is defined in (5). Given $n > 1$, let $k = k(n)$ be such that $q_k \leq n < q_{k+1}$. We have

$$\rho_n(x) \leq \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2} \leq \frac{K^2}{n^2}\,,$$

i.e., the right–hand side of (6) with $c_2 = K^2$. The left–hand side of (6) with $c_1 = c$ follows immediately from (5). □

3 Khinchin's theorem and proof of Theorem 3

3.1 Khinchin's metric theorem

Before starting the proof of Theorem 3 let us formulate Khinchin's metric theorem in the following standard form (see for instance [4, 5, 7]).

Khinchin's metric theorem [5]. *Let $F(\cdot)$ be a positive function on $[1, \infty)$ such that the function $xF(x)$ is decreasing. Then*
(i) *the inequality*

$$\left| x - \frac{p}{q} \right| < \frac{F(q)}{q} \tag{22}$$

has an infinite number of solutions in positive integers p, q for a.a. $x \in [0,1]$ if the integral

$$\int_1^\infty F(t)dt \tag{23}$$

diverges;
(ii) *the inequality* (22) *has only a finite number of solutions in positive integers p, q for a.a. $x \in [0,1]$ if the integral* (23) *converges.*

Set $f(t) = e^{-t}/F(e^t)$ for $t \geq 0$. Then the function F in Khinchin's metric theorem can be represented in the form $F(x) = 1/(xf(\log x))$ and therefore

$$\int_e^\infty F(t)dt = \int_1^\infty \frac{1}{f(t)}dt$$

The assumption for $xF(x)$ to be a decreasing function is equivalent to the assumption that the function $f(x)$ is increasing. Thus the assumptions in Khinchin's metric theorem agree with the conditions of Theorem 3.

3.2 Proof of Theorem 3

(i). Let the integral (7) diverge. Then according to Khinchin's metric theorem the inequality (22) holds for infinitely many q for a.a. $x \in [0,1]$. Setting $n = q$ we get

$$\rho_n(x) \le \left| x - \frac{p}{n} \right| \le \frac{1}{n^2 f(\log n)}$$

and therefore for a.a. $x \in [0,1]$ the inequality (8) holds for infinitely many integers n.

(ii). Let the integral (7) converge and

$$\rho_n(x) < \frac{1}{n^2 f(\log n)} \tag{24}$$

and let p/q be the fraction in $\mathcal{F}_n$ such that

$$\rho_n(x) = \left| x - \frac{p}{q} \right| .$$

Then using the fact that f is increasing we obtain

$$\frac{1}{q^2 f(\log q)} \ge \frac{1}{n^2 f(\log n)} \ge \left| x - \frac{p}{q} \right| = \rho_n(x) .$$

The proof is completed by showing that the inequality (24) holds for finitely many n for almost all irrationals $x \in [0,1)$. Indeed, the assumption that for a given x (24) holds for infinitely many n implies that the inequality

$$\frac{1}{q^2 f(\log q)} \ge \left| x - \frac{p}{q} \right|$$

has an infinite number of solutions in positive integers p, q and application of Khinchin's metric theorem leads to the required conclusion. □

4 Proof of Theorem 4

4.1 Auxiliary statements

The proof of Theorem 4 will use some tools developed in [7] in the course of proving Khinchin's metric theorem. We shall use two statements formulated in the form of lemmas.

Lemma 1 [7, Lemma 5, Ch. 1].
Let (Ω, Ξ, μ) be a measure space and let A_q be a sequence of sets in Ξ. Then if

$$\sum_{q=1}^{\infty} \mu(A_q) = \infty$$

then the set A of the points in Ω, which belong to an infinite number of sets A_q, has the measure

$$\mu(A) \geq \limsup_{m\to\infty} \frac{\left(\sum_{q=1}^{m} \mu(A_q)\right)^2}{\sum_{p=1}^{m}\sum_{q=1}^{m} \mu(A_p \bigcap A_q)}\,. \tag{25}$$

Proof is given in [7]. In what follows $\Omega = [0,1]$ and μ is the Lebesgue measure on $[0,1]$.

For $n \geq k \geq 1$ introduce the sets

$$A_{k,n} = \bigcup_{p:p<k,(p,k)=1} \left(\frac{p}{k} - \frac{1}{2nk}, \frac{p}{k} + \frac{1}{2nk}\right) \tag{26}$$

and note the obvious fact that the intervals in the union are not intersecting for fixed k and n.

The next lemma is not exactly the result of [7] although it is principally contained in the proof of Theorem 7, Ch. 1. We shall give a proof for the sake of completeness.

Lemma 2 *Let $1 \leq k \leq n$, $1 \leq l \leq m$ and $k \neq l$. Then*

$$|A_{k,n} \cap A_{l,m}| \leq \frac{4}{nm}\,. \tag{27}$$

Proof. Obviously, $|A_{k,n}| = \varphi(k)/(kn)$ and

$$|A_{k,n} \cap A_{l,m}| \leq \min\left\{\frac{1}{kn}, \frac{1}{ml}\right\} N(k,l,n,m) \tag{28}$$

where $N(k,l,n,m)$ is the number of pairs of positive integers p, p' such that

$$\left|\frac{p}{k} - \frac{p'}{l}\right| < \frac{1}{2kn} + \frac{1}{2ml}, \quad (p,k)=1,\ (p',l)=1,\ 0<p<k,\ 0<p'<l\,. \tag{29}$$

Let us derive an upper bound for $N(k,l,n,m)$.

Assume

$$pl - p'k = t \tag{30}$$

for some integer t. Then $d = (k,l)$ is a divisor of t and then setting

$$k = d\tilde{k},\ l = d\tilde{l},\ t = d\tilde{t}$$

we get $p\tilde{l} - p'\tilde{k} = \tilde{t}$, $(\tilde{l},\tilde{k}) = 1$. If $\tilde{p}$, $\tilde{p}'$ is another pair satisfying (30) then

$$p = \tilde{p} + s\tilde{k}, \quad \tilde{p} = \tilde{p}' + s\tilde{l} \tag{31}$$

where s is some integer. Assume $l < k$. Then we are interested in counting the number of pairs of integers $p, \tilde{p}$ falling into the interval $(0,k)$. Thus it should hold

$$|p - \tilde{p}| = |s|\tilde{k} < k = d\tilde{k}$$

which implies $|s| < d$. Therefore for fixed t and p the number of $\tilde{p}$ satisfying (30) is upper bounded by $2d - 1$ and using (31) we get that $2d - 1$ is also an upper bound for the number of pairs of integers p, $\tilde{p}$ for a fixed t. Finally, (29) gives

$$0 \neq |t| < \frac{l}{2n} + \frac{k}{2m}$$

and we can take only t such that d divides t. Altogether, this gives

$$N(k, l, n, m) \leq 2 \left\lfloor \frac{\frac{l}{2n} + \frac{k}{2m}}{d} \right\rfloor (2d - 1) < 4 \left(\frac{l}{2n} + \frac{k}{2m} \right) .$$

The required inequality (27) follows now from (28). □

4.2 Proof of Theorem 4

Let us start with prooving (ii). Mention first that the statement of the theorem is equivalent to the analogous statement with the logarithm on the base 2 substituted for the natural logarithm. We shall prove (ii) for this reformulation of the theorem.

Let the integral (7) converge and $x \in [0, 1]$. For any positive integer n define $k = k(n) = \lfloor \log_2 n \rfloor$, i.e., k is such that $2^k \leq n < 2^{k+1}$. If

$$\rho_n(x) \geq \frac{f(\log_2 n)}{n^2} , \tag{32}$$

that is, the inequality (11), with log replaced by $\log_2$, holds then due to monotonicity of f

$$\rho_{2^k}(x) \geq \rho_n(x) \geq \frac{1}{4} \frac{f(\log_2 2^k)}{(2^k)^2} = \frac{1}{4} \frac{f(k)}{(2^k)^2} .$$

Therefore if the inequality (32) holds for infinitely many n then the inequality

$$\rho_n(x) \geq \frac{1}{4} \frac{f(k)}{n^2} \qquad \text{with} \quad n = 2^k \tag{33}$$

holds for infinitely many k. This means that it is enough to prove (ii) only for the case when n goes through the sequence $n = 2^k$, $k = 1, 2, \ldots$.

Set

$$B_k = \{x \in [0, 1] \ \text{ such that the inequality (33) holds}\} .$$

Let us derive an upper bound for $|B_k|$, the Lebesgue measure of B_k, in order to apply the Borell–Cantelli lemma.

Consider the set $\mathcal{S}$ of all intervals I from the Farey partition $\mathcal{P}_n$ with $n = 2^k$ such that their length

$$|I| \geq \frac{1}{2} \frac{f(k)}{n^2} . \tag{34}$$

The union of these intervals contains B_k and therefore

$$|B_k| \le S = \sum_{I \in \mathcal{S}} |I| .$$

Compute an upper bound for S.

Let

$$I = \left[\frac{p}{q}, \frac{p'}{q'}\right) \in \mathcal{P}_n$$

and $a = \min\{q, q'\}$, $b = \max\{q, q'\}$. Then $b \ge n/2$ and thus $|I| = 1/(ab) \le 2/(an)$. Therefore for $I \in \mathcal{S}$

$$\frac{f(k)}{2n^2} \le |I| \le \frac{2}{an}$$

which particularly implies

$$a = \min\{q, q'\} \le \frac{4n}{f(k)} .$$

We thus have

$$|B_k| \le \sum_{I \in \mathcal{S}} |I| \le 2 \sum_{q=1}^{\lfloor \frac{4n}{f(k)} \rfloor} \varphi(q) \frac{2}{qn} \le \frac{4}{n} \sum_{q=1}^{\lfloor \frac{4n}{f(k)} \rfloor} 1 \le \frac{16}{f(k)}$$

where the factor 2 is due to that ≤ 2 intervals in $\mathcal{S}$ correspond to a fixed p/q with $q > 1$. We have also used here that for a given $q > 1$, $\varphi(q)$ is the number of fractions $p/q \in \mathcal{F}_n$ and that $\varphi(q) \le q$.

This leads to the inequality

$$\sum_{q=1}^{\infty} |B_k| \le 16 \int_1^{\infty} \frac{dt}{f(t)} + \frac{16}{f(1)} < \infty .$$

Applying the Borel–Cantelli arguments, we obtain (ii).

Turn to the proof of (i). Let for some $\beta > 0$

$$\limsup_{t \to \infty} \frac{\log f(t)}{t} < \frac{\beta}{1+\beta}$$

where the existence of β is guaranteed by the assumption (10).

For all integers $k \ge 1$ define $n_k = \lfloor 6kf((1+\beta)\log k) \rfloor + 1$ and

$$B_k = \bigcup_{p: p<k, (p,k)=1} \left(\frac{p}{k} + \frac{1}{6kn_k}, \frac{p}{k} + \frac{1}{3kn_k} \right) \tag{35}$$

Introduce also the set

$$B = \{x \in [0,1] \text{ such that the inequality (11) holds infinitely often}\} .$$

Regularity condition (10) implies there exists k' such that for $k \geq k'$

$$(1+\beta)\log k \geq \log 7 + \log k + \log f((1+\beta)\log k)$$

and thus

$$(1+\beta)\log k \geq \log(6kf((1+\beta)\log k)+1)\,.$$

Since f is increasing we deduce

$$f((1+\beta)\log k) \geq f(\log n_k)\,.$$

Finally for $x \in B_k$

$$\rho_{n_k}(x) \geq \frac{1}{6kn_k} = \frac{n_k}{6kn_k^2} \geq \frac{6kf((1+\beta)\log k)}{6kn_k^2} \geq \frac{f(\log n_k)}{n_k^2}\,.$$

Thus every $x \in B_k$ satisfies (11) for $k \geq k'$.

Hence it will suffice to prove that $|B| = 1$.

Applying Lemma 1 we get

$$|B| \geq \limsup_{m\to\infty} \frac{\left(\sum_{k=1}^m |B_k|\right)^2}{\sum_{k=1}^m \sum_{l=1}^m \mu(B_k \cap B_l)}\,. \tag{36}$$

Let us first construct a lower bound for the numerator in (36):

$$|B_k| = \frac{1}{6}|A_{k,n_k}| = \frac{\varphi(k)}{6kn_k}$$

where the sets A_{k,n_k} are defined in (26), and therefore

$$\sum_{k=1}^m |B_k| \geq \frac{1}{6}\sum_{k=1}^m \frac{\varphi(k)}{kn_k}\,.$$

Applying Abel transformation we get

$$\sum_{k=1}^m \frac{\varphi(k)}{kn_k} = \sum_{k=1}^m (N(k)-N(k-1))\frac{1}{kn_k} =$$

$$= \sum_{k=1}^{m-1} N(k)\left(\frac{1}{kn_k} - \frac{1}{(k+1)n_{k+1}}\right) + N(m)\frac{1}{mn_m} \geq$$

$$\geq \frac{1}{2}\sum_{k=1}^{m-1} \frac{k(k+1)}{2}\left(\frac{1}{kn_k} - \frac{1}{(k+1)n_{k+1}}\right) + \frac{m(m+1)}{4}\frac{1}{mn_m} = \frac{1}{2}\sum_{k=1}^m \frac{1}{n_k}$$

where

$$N(0) = 0, \quad N(k) = \sum_{l=1}^k \varphi(l) \geq \frac{k(k+1)}{4}$$

for all $k \geq 1$.

Therefore

$$\sum_{k=1}^{m} |B_k| \geq \frac{1}{12} \sum_{k=1}^{m} \frac{1}{n_k}\,.$$

Let us demonstrate that

$$\sum_{k=1}^{\infty} \frac{1}{n_k} = +\infty\,. \tag{37}$$

Indeed, there exists $k_0 > 0$ such that $n_k \leq 7kf((1+\beta)\log k)$ for all $k \geq k_0$ and therefore

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \geq \frac{1}{7} \sum_{k=k_0}^{\infty} \frac{1}{kf((1+\beta)\log k)} \geq \frac{1}{7} \int_{k_0+1}^{\infty} \frac{dt}{tf((1+\beta)\log t)} =$$

$$= \frac{1}{7} \int_{\log(k_0+1)}^{\infty} \frac{d\tau}{f((1+\beta)\tau)} = \infty\,.$$

Let us turn to the denominator in (36) and estimate $|B_k \cap B_l|$ for $k \neq l$. According to Lemma 2 for $k \neq l$

$$|B_k \cap B_l| \leq |A_{k,n_k} \cap A_{l,n_l}| \leq \frac{4}{n_k n_l}$$

and therefore

$$\sum_{k,l=1}^{m} |B_k \cap B_l| = 2 \sum_{1 \leq k < l \leq m} |B_k \cap B_l| + \sum_{k=1}^{m} |B_k| \leq$$

$$\leq 8 \sum_{1 \leq k < l \leq m} \frac{1}{n_k n_l} + \sum_{k=1}^{m} \frac{1}{n_k} \leq 4 \sum_{k=1}^{m} \left(\sum_{k=1}^{m} \frac{1}{n_k} \right)^2 + \sum_{k=1}^{m} \frac{1}{n_k}\,.$$

Using (37) we get

$$\lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{1}{n_k} \right) \Big/ \left(\sum_{k=1}^{m} \frac{1}{n_k} \right)^2 = 0\,.$$

Therefore

$$|B| \geq \limsup_{m\to\infty} \frac{\left(\frac{1}{12} \sum_{k=1}^{m} (1/n_k) + O(1) \right)^2}{4 \left(\sum_{k=1}^{m} (1/n_k) \right)^2 + \sum_{k=1}^{m} (1/n_k)} = \gamma = \frac{1}{576}\,.$$

Let us now show that for all integers $m \geq 2$, $0 \leq l \leq m-1$,

$$\left| B \bigcap \left(\frac{l}{m}, \frac{l+1}{m} \right) \right| \geq \frac{\gamma}{m}\,.$$

Indeed, let us construct a set B_1 analogous to B and corresponding to the function $f_1(\cdot) = mf(\cdot)$. Then the above yields $|B_1| \geq \gamma$. Let $x \in B_1$ then it is easy to see that $x' = (x+l)/m \in B$. As the matter of fact, if

$$\rho_n(x') \leq \frac{f(\log n)}{n^2}$$

and $\rho_n(x') = |x' - p/q|$ for some

$$\frac{p}{q} \in \mathcal{F}_n \bigcap \left(\frac{l}{m}, \frac{l+1}{m}\right)$$

then

$$\left| x - \frac{pm - l}{q} \right| \leq \frac{mf(\log n)}{n^2} = \frac{f_1(\log n)}{n^2}$$

and $0 < pm - l < q$, therefore if the inequality (11) holds for x with $f_1(\cdot) = mf(\cdot)$ then it also holds for x' with f.

To complete the proof we only need to apply Lebesgue theorem on the density points, see for example Ch. 11 in [8], and obtain $|B| = 1$. $\square$

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