Asymptotic distribution of the distance function to the Farey points

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Abstract

Let x be a real number in $[0,1]$, $\mathcal{F}_n$ be the Farey sequence of order n and $\rho_n(x)$ be the distance between x and $\mathcal{F}_n$. Assuming that $n \to \infty$ we derive the asymptotic distributions of the functions $n^2\rho_n(x)$ and $n\rho_n(x'/n),\ 0 \leq x' \leq n$. We also establish the asymptotics for $\int_0^1 \rho_n^\delta(x)dx$, for all real δ.

Key words: Farey sequence, rational approximation, diophantine approximation, asymptotic distribution.

1 Introduction: Statement of the problem and formulation of the main results

Let x be a real number in $[0,1]$ and $\mathcal{F}_n$ be the Farey sequence of order n, that is, the collection of all rationals p/q with $p \leq q$, $(p,q)=1$ and the denominators $q \leq n$. In the present work we derive two asymptotic distributions for

$$\rho_n(x) = \min_{p/q \in \mathcal{F}_n} \left| x - \frac{p}{q} \right| ,$$

the distance function between x and $\mathcal{F}_n$, and establish the asymptotics for $\int_0^1 \rho_n^\delta(x)dx$, for all real δ.

It is well–known that the elements of the Farey sequence $\mathcal{F}_n$ are uniformly distributed asymptotically, when $n \to \infty$, and this has important consequences in number theory: for example, the Riemann hypothesis can be formulated in terms of the rate of convergence of $\mathcal{F}_n$ to the uniform distribution, see [1, 2, 3]. However, little is known about other asymptotic properties of $\mathcal{F}_n$ and the distance function $\rho_n(x)$.

In our previous work [4] we have established some metric theorems concerning $\rho_n(x)$. Specifically, we have shown that for suitable functions $f(\cdot)$ the inferior and superior limits,

$$\liminf_{n\to\infty} n^2\rho_n(x)f(\log n) \ \text{ and } \ \limsup_{n\to\infty} n^2\rho_n(x)/f(\log n)\,,$$

may achieve only values 0 and ∞, for almost all x with respect to the Lebesgue measure on [0,1], depending on whether $\int_1^\infty dx/f(x)$ converges or diverges. In the present work we continue the study of the asymptotic behaviour of the distance function $\rho_n(x)$. The main results of the paper are formulated in the following four theorems.

Theorem 1.1. *The sequence of functions*

$$\tilde{\rho}_n(x') = \begin{cases} n\rho_n(x'/n) & \text{if } 0 \le x' \le n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

converge in distribution, when $n \to \infty$, *to the measure* $\tilde{p}(\tau)d\tau$ *on* $\mathcal{B}$ *with the density*

$$\tilde{p}(\tau) = \begin{cases} 2\sum_{k=1}^{m} \varphi(k) & \text{for } \tau \in \left(\frac{1}{2(m+1)}, \frac{1}{2m}\right], \quad m = 1, 2, \ldots \\ 0 & \text{for } \tau \notin (0, \frac{1}{2}] \end{cases} \tag{2}$$

that is, for any a, A *such that* $0 < a < A < \infty$,

$$n \,\mathrm{meas}\{x \in [0,1] : \ a < n\rho_n(x) \le A\} \to \int_a^A \tilde{p}(\tau)d\tau, \quad n \to \infty.$$

Here and in what follows 'meas' stands for the Lebesgue measure on $[0,1]$, $\mathcal{B}$ denotes the σ–algebra of the Borel subsets of $(0,\infty)$ and $\varphi(\cdot)$ is the Euler function.

Theorem 1.2. *The sequence of functions* $n^2\rho_n(x)$ *converge in distribution, when* $n \to \infty$, *to the probability measure* $\hat{p}(\tau)d\tau$ *on* $\mathcal{B}$ *with the density*

$$\hat{p}(\tau) = \begin{cases} 6/\pi^2 & \text{if } 0 \le \tau \le \frac{1}{2} \\ \frac{6}{\pi^2\tau}\left(1 + \log\tau - \tau\right) & \text{if } \frac{1}{2} \le \tau \le 2 \\ \frac{3}{\pi^2\tau}\left(2\log(2\tau) - 4\log(\sqrt{\tau} + \sqrt{\tau-2}) - (\sqrt{\tau} - \sqrt{\tau-2})^2\right) & \text{if } 2 \le \tau < \infty \end{cases} \tag{3}$$

that is, for any a, A *such that* $0 < a < A < \infty$,

$$\mathrm{meas}\{x \in [0,1] : \ a < n^2\rho_n(x) \le A\} \to \int_a^A \hat{p}(\tau)d\tau, \quad n \to \infty.$$

One of the key elements in the proof of Theorem 1.2 is the asymptotic two-dimensional uniformity of the pairs of the denominators of the neighbours in the Farey sequences. Specifically, the following result holds.

Let p/q and p'/q' be neighbours in $\mathcal{F}_n$ such that $0 \le p/q < p'/q' \le 1$. The ordered pair (q, q') will be called the neighbouring pair of denominators in $\mathcal{F}_n$, the number of such pairs equals $N(n) = |\mathcal{F}_n| - 1 = \sum_{q=1}^{n} \varphi(q)$. Let ν_n be the two-variate probability measure assigning the mass $1/N(n)$ to each pair $(q/n, q'/n)$ where (q, q') take all possible values in the set of all neighbouring pairs of denominators in $\mathcal{F}_n$.

Theorem 1.3. *The sequence of probability measures* ν_n *weakly converge, when* $n \to \infty$, *to the uniform probability measure on the triangle* $\{(x, y) : 0 \le x, y \le 1, x + y \ge 1\}$.

An important result, which is essentially a consequence of Theorems 1.1 and 1.2, concerns the asymptotic behaviour of the moments of the distance function $\rho_n(x)$.

Theorem 1.4. *For any $\delta \neq 0$ and $n \to \infty$*

$$\frac{\delta+1}{2}\int_0^1 \rho_n^\delta(x)dx = \begin{cases} \infty & \text{if } \delta \le -1 \\ \frac{3}{\delta^2\pi^2}\left(2^{-\delta} + \delta 2^{\delta+2}\mathrm{B}(-\delta, \tfrac{1}{2})\right) n^{-2\delta}\,(1+o(1)) & \text{if } -1<\delta<1, \delta\neq 0 \\ \frac{3}{\pi^2} n^{-2}\log n + O\left(n^{-2}\right) & \text{if } \delta = 1 \\ 2^{-\delta}\frac{\zeta(\delta)}{\zeta(\delta+1)} n^{-\delta-1} + O\left(n^{-2\delta}\right) & \text{if } \delta > 1 \end{cases} \tag{4}$$

where $\zeta(\cdot)$ and $B(\cdot,\cdot)$ are the Riemann zeta–function and the Beta–function, correspondingly.

The paper is organized as follows. In Section 2 we formulate and prove a number of technical lemmas that are used in the proofs of the main theorems. All statements of Section 2 are of a general character, for instance, the notion of the Farey sequence is used in neither of these statements.

Section 3 is devoted to the study of the asymptotic distribution of the sequence of functions (1). In this section we prove Theorem 3.1 which includes, as particular cases, Theorem 1.1 and a part of Theorem 1.4.

In Section 4 we study the asymptotic distribution of two sequences of probability measures associated with the functional sequence $n^2\rho_n(x)$. In this section we prove Theorem 1.3 and Theorem 4.1, the latter includes Theorem 1.2 as a component.

Theorem 1.4 is a corollary of three theorems, specifically, Theorem 3.1, in the case $\delta > 1$, Theorem 4.1, the case $-1 < \delta < 1$, and Theorem 1 in [4], the case $\delta = 1$.

2 Auxiliary results

In this section we prove several simple technical lemmas which shall be used in the next sections. First, we introduce some notation.

Let $\mathcal{B}$ be the σ–algebra of Borel subsets of $(0,\infty)$ and $\mathcal{M}$ be the set of the Borel measures on $\mathcal{B}$, these measures attach finite values to all intervals $[a, A]$ with $0<a<A<\infty$.

We shall say that a sequence of measures μ_n in $\mathcal{M}$ $*$-weakly converge to a measure $\mu \in \mathcal{M}$ and write $\mu_n \xrightarrow{*} \mu$, $n \to \infty$, if μ_n converge to μ in the sense of the theory of Schwartz's distributions. That is, $\mu_n \xrightarrow{*} \mu$ when $n \to \infty$, if for any continuous function g on $(0,\infty)$ with compact support $\int g d\mu_n \to \int g d\mu$. In the other words, $\mu_n \xrightarrow{*} \mu$, $n \to \infty$, if for any $0<a<A<\infty$ $\mu_n|_{(a,A)} \Longrightarrow \mu|_{(a,A)}$, $n \to \infty$, in the usual sense of weak convergence of finite measures, see [5]. For a thorough description of the $*$-weak convergence of measure sequences see, for example, [6], ch.6.

In the first two lemmas of this section we establish a general relation between two measures: the first one is the distribution of the distance function $\rho_n(x)$ and the other assigns equal masses to all interval lengths of the partition generated by $\mathcal{F}_n$. This relation does not depend on the particular form of the Farey sequences and we thus consider a more general case.

For every $n = 1, 2, \ldots,$ let $N(n)$ be a positive integer and $\mathcal{F}_n$ be an ordered collection of $N(n)+1$ points in [0,1]:

$$\mathcal{F}_n = \{x_{0,n}, x_{1,n}, \ldots, x_{N(n),n} :\ 0 = x_{0,n} < x_{1,n} < \ldots < x_{N(n),n} = 1\} \tag{5}$$

With every point collection $\mathcal{F}_n$ of this kind we associate the partition $\mathcal{P}_n$ of [0,1):

$$\mathcal{P}_n :\quad [0,1) = \bigcup_{i=1}^{N(n)} I_{i,n} \quad \text{where} \quad I_{i,n} = [x_{i-1,n}, x_{i,n})\,, \tag{6}$$

and the collection of interval lenghts:

$$\{p_{i,n} = |I_{i,n}| = x_{i,n} - x_{i-1,n},\ \ i = 1, \ldots, N(n)\}\,. \tag{7}$$

In Sections 3 and 4, when $\mathcal{F}_n$ will stand for the Farey sequence, $\mathcal{P}_n$ will go under the name of the Farey partition.

For every n, let us define the measure $\mu_n \in \mathcal{M}$ by assigning the mass 1 to the points $p_{i,n},\ i = 1, \ldots, N(n)$. We write this measure as

$$\mu_n = \sum_{i=1}^{N(n)} \delta(t - p_{i,n}) \tag{8}$$

where $\delta(\cdot)$ is the Dirac delta function.

For two numerical sequences of positive normalization constants F_n and G_n we also define the normalized measures $\mu_n(F_n, G_n)$ by assigning equal masses G_n to the points $F_n p_{i,n}$, $i = 1, \ldots, N(n)$:

$$\mu_n(F_n, G_n) = \sum_{i=1}^{N(n)} G_n \delta(t - F_n p_{i,n})\,. \tag{9}$$

In a particular case, when $F_n = G_n = 1,\ \ \mu_n(1,1) = \mu_n$. Note also that for all n and sequences of positive constants F_n and G_n the measures $\mu_n(F_n, G_n)$ are defined on $\mathcal{B}$. We will be interested in the sequences $\{F_n\}$ and $\{G_n\}$ which provide the $*$-weak convergence, when $n \to \infty$, of the sequence $\{\mu_n(F_n, G_n)\}_n$ to certain non-degenerate Borel measures μ on $\mathcal{B}$. Since $\int_0^\infty d\mu_n(F_n, G_n) = N(n)G_n$, we do not necessary expect that the limit measures are finite, that is $\mu((0,\infty)) < \infty$.

For any $x \in [0,1]$ consider the distance between x and $\mathcal{F}_n$:

$$\rho_n(x) = \rho(x, \mathcal{F}_n) = \min_{x_{i,n} \in \mathcal{F}_n} |x - x_{i,n}|\ .$$

This is a measurable function, with respect to the σ-algebra of Borel subsets of $[0,1]$, and it can be associated with the probability measure $d\Phi_n(t)$ where

$$\Phi_n(t) = \text{meas}\,\{x \in [0,1] :\ \rho_n(x) \le t\}\,. \tag{10}$$

The following statement shows that there exists a simple relationship between the measure μ_n, defined in (8), and the density corresponding to (10).

Lemma 2.1. *Let $n \geq 1$, $N(n) \geq 1$ and $\mathcal{F}_n$ be any collection of points* (5)*. Then the measure $d\Phi_n$ is absolutely continuous with respect to the Lebesgue measure, its density $p_n(\tau) = \Phi_n'(\tau)$ is such that $p_n(\tau) = 0$ for $\tau \notin [0, \frac{1}{2}]$ and*

$$p_n(\tau) = 2\mu_n((2\tau, +\infty)) = 2 \sum_{i: p_{i,n} > 2\tau} 1 \quad \text{for any } \tau > 0 \tag{11}$$

where the measure μ_n is defined in (8) *and $p_{i,n}$ are defined in* (7)*.*

Proof. We have for any n and $\tau > 0$:

$$1 - \Phi_n(\tau) = \text{meas}\{x \in [0,1] :\ \rho_n(x) > \tau\} = \sum_{i=1}^{N(n)} \text{meas}\{x \in I_i,\ \rho_n(x) > \tau\}$$

$$= \sum_{i: |I_i| > 2\tau} \text{meas}\{x \in I_i,\ \rho_n(x) > \tau\}$$

$$= 2 \sum_{i: |I_i| > 2\tau} \text{meas}\{x \in \left[x_{i-1,n}, \frac{x_{i-1,n} + x_{i,n}}{2}\right),\ \rho_n(x) > \tau\}$$

$$= 2 \sum_{i: p_{i,n} > 2\tau} \text{meas}\{x \in [x_{i-1,n}, x_{i-1,n} + p_{i,n}/2)\,,\ x - x_{i-1,n} > \tau\} = 2 \sum_{i: p_{i,n} > 2\tau} (p_{i,n}/2 - \tau)$$

$$= 2 \sum_{i: p_{i,n} > 2\tau} \int_\tau^{p_{i,n}/2} 1 dt = 2 \int_\tau^\infty \sum_{i: p_{i,n} > 2t} 1 dt = 2 \int_\tau^\infty \mu_n((2t, \infty)) dt$$

This implies that the measure $d\Phi_n(t)$ is absolutely continuous, with respect to the Lebesgue measure, and it also yields the validity of the relation (11). The fact that $p_n(\tau) = 0$ for $\tau \notin [0, \frac{1}{2}]$ follows from the definition of p_n. □

The following statement is an obvious consequence of Lemma 2.1.

Corollary 2.1. *For any two positive sequences $\{F_n\}$ and $\{G_n\}$*

$$p_n(\tau/F_n) G_n = 2 G_n \mu_n((2\tau/F_n, \infty)) = 2 \sum_{i: p_{i,n} F_n > 2\tau} G_n \quad \text{for any } \tau > 0 \tag{12}$$

where the density $p_n(\cdot)$ and the measure μ_n are the same as in Lemma 2.1.

Lemma 2.2. *Let the sequence of partitions $\{\mathcal{P}_n\}$ of $[0,1)$ and the numerical sequences $\{F_n\}$, $\{G_n\}$ be such that the sequence of measures $\{\mu_n = \mu_n(F_n, G_n)\}_n$ defined through*

(9) $*$*-weakly converge, when* $n \to \infty$*, to some Borel measure* μ *and for some given* A*, a point of continuity of the measure* μ*,*

$$\mu_n([A,\infty)) \to \mu([A,\infty)) < \infty, \quad n \to \infty\,. \tag{13}$$

Then the sequence of measures $\{p_n(\tau/F_n)G_n d\tau\}$ $*$*-weakly converge to an absolutely continuous, with respect to the Lebesgue measure on* $(0,\infty)$*, measure* $p(\tau)d\tau$ *where*

$$p(\tau) = 2\mu([2\tau,\infty)) \tag{14}$$

for any $\tau > 0$ *such that* 2τ *is the point of continuity of the measure* μ*. Besides, the sequence of functions* $\{p_n(\tau/F_n)G_n\}$ *converge to* $p(\tau)$ *for all such* τ*.*

Proof. Let the sequence of measures $\{\mu_n = \mu_n(F_n, G_n)\}_n$ $*$-weakly converge, when $n\to\infty$, to some Borel measure μ and $\mu_n([A,\infty)) \to \mu([A,\infty)) < \infty$ for some A, a point of continuity of the measure μ. Let B be any point of continuity of the measure μ and let, say, $0 < B < A$. Then $\mu_n([B,\infty)) = \mu_n([B,A)) + \mu_n([A,\infty))$. Using (13) and the fact that $*$-weak convergence of measures on open intervals coincides with the standard weak convergence, we get

$$\lim_{n\to\infty} \mu_n([B,\infty)) = \lim_{n\to\infty} \mu_n([B,A)) + \lim_{n\to\infty} \mu_n([A,\infty)) = \mu([B,\infty))\,. \tag{15}$$

The relation (15) can be analogously proven for $B \geq A$ and it thus holds for any B, the point of continuity of the measure μ. The relations (12) and (15) yield

$$p_n(\tau/F_n)G_n = 2G_n\mu_n((2\tau/F_n,\infty)) \to 2\mu([2\tau,\infty)) = p(\tau), \;\; n \to \infty\,,$$

for any $\tau > 0$ such that 2τ is the point of continuity of the measure μ.

Let us now fix τ_1 and τ_2 such that $0<\tau_1<\tau_2$ and $2\tau_1$, $2\tau_2$ are the points of continuity of the measure μ. Since $p_n(\tau/F_n)G_n$ is monotonously decreasing with respect to τ, $p_n(\tau/F_n)G_n \leq p_n(\tau_1/F_n)G_n$ for any $\tau \in [\tau_1,\tau_2]$ and therefore according to the Lebesgue theorem on the dominated convergence for any such τ_1 and τ_2

$$\int_{\tau_1}^{\tau_2} p_n(\tau/F_n)G_n d\tau \to \int_{\tau_1}^{\tau_2} p(\tau)d\tau\,, \quad n \to \infty\,.$$

This completes the proof. □

Lemma 2.3. *Let a measure* μ *and a function* p *be related via (14) and* $\int_0^\infty t^{\delta+1}d\mu(t)<\infty$ *for some real* δ*. Then*

$$\int_0^\infty t^\delta p(t)dt = \begin{cases} +\infty & \text{if } \delta \leq -1 \\ C_\delta \int_0^\infty t^{\delta+1}d\mu(t) < +\infty & \text{if } \delta > -1 \end{cases} \tag{16}$$

where

$$C_\delta = \frac{1}{(1+\delta)2^\delta}\,. \tag{17}$$

Proof. Using (14) and the Fubini theorem, we get for any $\delta > -1$:

$$\int_0^\infty \tau^\delta p(\tau)d\tau = \int_0^\infty \tau^\delta 2\mu([2\tau,\infty))d\tau = \int_0^\infty 2\tau^\delta \int_{2\tau-}^\infty d\mu(t)d\tau =$$

$$2\int_0^\infty \int_0^{t/2} \tau^\delta d\tau d\mu(t) = C_\delta \int_0^\infty t^{\delta+1} d\mu(t)\,.$$

If $\delta \le -1$ then one of the integrals in the chain, namely $\int_0^{t/2} \tau^\delta d\tau$, diverges; this yields that the first integral in the chain also diverges. □

Lemmas 2.2 and 2.3 establish a correspondence between the asymptotic behaviour of the distributions of the functions $\rho_n(x) = \rho(x, \mathcal{F}_n)$ and the distributions μ_n of the interval lengths of the partitions generated by $\mathcal{F}_n$, as well as a relation between the moments of these distributions. The next problem is to find a convenient sufficient condition for the convergence, when $n \to \infty$, of a properly normalized sequence of measures $\{\mu_n\}$.

Let us associate with every $\mu \in \mathcal{M}$ its Mellin transform

$$M(\mu)(s) = \int_0^\infty t^s d\mu(t) \tag{18}$$

which is defined and analytic in the strip $\{s : \operatorname{Re} s \in (A,B)\}$ where (A,B) is the biggest open interval such that $\int_0^\infty t^\alpha d\mu(t) < \infty,\ \ \alpha \in (A,B)$. According to the S.N.Bernstein theorem, see [7], the set $W_{a,b}$ of functions f on (a,b) which can be represented in the form $f = M(\mu)|_{(a,b)},\ \ \mu \in \mathcal{M}$, can be also described as follows: $f \in W_{a,b}$ if and only if f is continuous and all forms $\sum_{i,k=1}^n f(x_i x_k)\xi_i\xi_k,\ n \ge 1$, such that $x_i x_k \in (a,b)$, are nonnegative.

For any $f \in W_{a,b}$, denote the measure in $\mathcal{M}$, corresponding to f, by $\mu(f)$. The following technical lemma relates the pointwise convergence of functions in $W_{a,b}$ and the $*$-weak convergence of the corresponding measures.

Lemma 2.4.
1. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $W_{a,b}$, $0<a<b<\infty$, and $f_n(x) \to f(x)$ for all $x \in (a,b)$. Then $f \in W_{a,b}$ and $\mu(f_n) \xrightarrow{*} \mu(f)$, $n \to \infty$. Besides, $M(\mu_n)$ converge to $M(\mu)$ uniformly on all compact subsets of the strip $\{s : \operatorname{Re} s \in (a,b)\}$.*
2. *Let $\{\mu_n\}$ be a sequence of measures in $\mathcal{M}$, $\mu_n \xrightarrow{*} \mu$ when $n \to \infty$, for some $0<a<b<\infty$*

$$\sup_{n\ge1} \int_0^\infty (t^a + t^b)d\mu_n(t) < +\infty\,, \tag{19}$$

and for some $\alpha \in (a,b)$

$$\int_0^\infty t^\alpha d\mu_n(t) \to \int_0^\infty t^\alpha d\mu(t)\,, \quad n \to \infty\,. \tag{20}$$

Then $M(\mu_n)(x) \to M(\mu)(x) < \infty,\ n \to \infty,\ $ for all $x \in (a,b)$.

Proof. 1. Let $\mu_n \in \mathcal{M}$ be such that $M(\mu_n) = f_n$, $n \geq 1$, and denote $H_n(s) = M(\mu_n)(s)$, $\operatorname{Re} s \in (a,b)$. Then for any a_1, b_1, such that $a < a_1 < b_1 < b$, the absolute values of the functions H_n, $n \geq 1$, are upper bounded by $\sup_{n\geq 1}(f_n(a_1) + f_n(b_1))$. Therefore, according to the Vitali theorem, see for example Theorem 5.2.1 in [8], the sequence of analytic functions $\{H_n(s)\}_n$ converge to some function $H(s)$ uniformly on compact subsets of the strip $\{s : \operatorname{Re} s \in (a,b)\}$. This implies that $f = H|_{(a,b)}$ is a continuous function and, moreover, according to the S.N.Bernstein theorem, see [7], $f \in W_{a,b}$ and therefore $f = M(\mu)$ for some $\mu \in \mathcal{M}$ and $H(s) = M(\mu)(s)$ for s such that $\operatorname{Re} s \in (a,b)$.

Let us fix some $\alpha \in (a,b)$ and consider the measures $d\lambda_n(t) = t^\alpha d\mu_n(t)$, $d\lambda(t) = t^\alpha d\mu(t)$. Then $\lambda_n((0,\infty)) = f_n(\alpha) \to f(\alpha) = \lambda((0,\infty))$, $n \to \infty$, and for every real y

$$\int_0^\infty t^{iy} d\lambda_n(t) = H_n(\alpha + iy) \longrightarrow H(\alpha + iy) = \int_0^\infty t^{iy} d\lambda(t)\,, \quad n \to \infty\,.$$

Using the standard existence criterion of the weak limit, we get the weak convergence $\lambda_n \Longrightarrow \lambda$ and therefore $\mu_n \xrightarrow{*} \mu$ when $n \to \infty$.

2. Let $\mu_n \in \mathcal{M}$, $\mu_n \xrightarrow{*} \mu$ when $n \to \infty$, and let H_n, H, λ_n and λ have the same meaning as above. Then, applying the well-known theorem of continuity, see e.g.[5], we get

$$\int_0^\infty t^{\alpha+iy} d\lambda_n(t) \to \int_0^\infty t^{\alpha+iy} d\lambda(t)\,, \;\; y \in R, \quad n \to \infty\,.$$

Besides, according to the proof of the first part of Lemma, H_n and H are uniformly bounded within the strip $\{s : \operatorname{Re} s \in (a,b)\}$. Therefore the Vitali theorem gives that $M(\mu_n)$ converge to $M(\mu)$, when $n \to \infty$, uniformly on compacts in the strip $\{s : \operatorname{Re} s \in (a,b)\}$. □

Lemma 2.5. *Let* $\mathbf{T}$ *be the unit circle,*

$$I_\alpha = \{e^{i\psi},\ -\alpha \leq \psi \leq \alpha\} \subseteq \mathbf{T}\,, \quad 0 \leq \alpha \leq \pi\,.$$

and let $\{\mu_n\}_{n=1}^\infty$ *be a sequence of probability measures on the unit circle* $\mathbf{T}$ *weakly converging to* m*, the normalized Lebesgue measure on* $\mathbf{T}$*. Then*

$$\lim_{n\to\infty} \mu_n(I_\alpha e^{i\phi}) = \alpha/\pi = m(I_\alpha)$$

uniformly with respect to $\phi \in [0, 2\pi)$.

Proof. Let an integer $n \geq 1$ be such that $1/n < \alpha < \pi - 1/n$. Consider functions $f_n, g_n \in C(\mathbf{T})$ such that $0 \leq f_n(\zeta) \leq 1$, $0 \leq g_n(\zeta) \leq 1$,

$$f_n(\zeta) = \begin{cases} 1 & \text{if } \zeta \in I_\alpha \\ 0 & \text{if } \zeta \notin I_{\alpha+1/n} \end{cases}\,, \quad g_n(\zeta) = \begin{cases} 1 & \text{if } \zeta \in I_{\alpha-1/n} \\ 0 & \text{if } \zeta \notin I_\alpha\,. \end{cases}$$

Then the families of functions

$$\{f_{n,\phi}(\zeta) = f_n(\zeta e^{i\phi}),\ \zeta \in \mathbf{T},\ \phi \in [0,2\pi]\}\,, \;\; \{g_{n,\phi}(\zeta) = g_n(\zeta e^{i\phi}),\ \zeta \in \mathbf{T},\ \phi \in [0,2\pi]\}$$

are compact sets in $C(\mathbf{T})$, since the former, for example, is the image of $\mathbf{T}$ for the continuous mapping $\phi \to f_{n,\phi}$ of the interval $[0, 2\pi]$ into $C(\mathbf{T})$. Since the point-wise convergence of linear functionals with the norm 1 yields the uniform convergence on compact subsets, we get

$$\lim_{k\to\infty} \int_{\mathbf{T}} f_{n,\phi}(\zeta) d\mu_k(\zeta) = \int_{\mathbf{T}} f_{n,\phi}(\zeta) d\mu(\zeta)\,, \quad \lim_{k\to\infty} \int_{\mathbf{T}} g_{n,\phi}(\zeta) d\mu_k(\zeta) = \int_{\mathbf{T}} g_{n,\phi}(\zeta) d\mu(\zeta)$$

uniformly with respect to $\phi \in [0, 2\pi]$. Besides, it is obvious that

$$\int_{\mathbf{T}} g_{n,\phi}(\zeta) d\mu_k(\zeta) \le \mu_k(I_\alpha e^{i\phi}) \le \int_{\mathbf{T}} f_{n,\phi}(\zeta) d\mu_k(\zeta)\,,$$

and

$$\int_{\mathbf{T}} g_{n,\phi}(\zeta) dm(\zeta) \le m(I_\alpha e^{i\phi}) \le \int_{\mathbf{T}} f_{n,\phi}(\zeta) dm(\zeta)$$

The transition to the limit yields the required. □

Finally, let us formulate a statement which may well be hidden in manuals on elementary probability theory.

Lemma 2.6. *Let α and β be independent random variables uniformly distributed on $[0, 1]$ and $t \ge 0$. Then the probability of the event $\{\alpha\beta \le t\}$, conditionally on $\alpha+\beta \ge 1$, equals* $\Pr\{\alpha\beta \le t | \alpha + \beta \ge 1\} =$

$$F(t) = \begin{cases} -4t \log \frac{1+\sqrt{1-4t}}{2} - \frac{(1-\sqrt{1-4t})^2}{2} & \text{if } \ 0 \le t \le \frac{1}{4} \\ 2t(1 - \log t) - 1 & \text{if } \ \frac{1}{4} \le t \le 1 \\ 1 & \text{if } \ t \ge 1 \end{cases} \tag{21}$$

and the moments of the probability measure $dF(t)$ exist for any $\alpha > -2$ and equal

$$M_\alpha = \int_0^1 t^\alpha dF(t) = \begin{cases} \frac{2}{\alpha+1}\left(\frac{1}{\alpha+1} - 4^{-\alpha}\mathrm{B}(\alpha+1, \frac{1}{2})\right) & \text{for } \alpha > -2,\ \alpha \ne -1 \\ \pi^2/3 & \text{for } \alpha = -1 \end{cases} \tag{22}$$

Proof. The proof is an exercise in calculation of integrals. The derivation of the formula (21) for $F(t)$ is easy. To derive (22) we have used integration by parts, the formula $(1-\sqrt{1-4t})(1+\sqrt{1-4t}) = 4t$, the representation $F(t) = F_1(t) - F_2(t)$ for $0 \le t \le \frac{1}{4}$ where

$$F_1(t) = -4t \log \frac{1+\sqrt{1-4t}}{2}\,, \quad F_2(t) = \frac{(1-\sqrt{1-4t})^2}{2}\,,$$

and the analytic expression for the following integral

$$I = \int_0^{1/4} t^\alpha \frac{1-\sqrt{1-4t}}{\sqrt{1-4t}} dt = \begin{cases} 4^{\alpha-1}\left(\mathrm{B}(\alpha+1, \frac{1}{2}) - \frac{1}{\alpha+1}\right) & \text{for } \alpha > -2,\ \alpha \ne -1 \\ \pi^2/3 & \text{for } \alpha = -1\,. \end{cases}$$

The formula (22) for $\alpha \neq -1$ follows then from

$$\int_0^{1/4} t^\alpha dF_1(t) = \frac{1}{\alpha+1}\left(4^{-\alpha}\log 2 + 2\alpha I\right), \quad \int_0^{1/4} t^\alpha dF_2(t) = 2I,$$

$$\int_{1/4}^1 t^\alpha dF(t) = \frac{2}{(\alpha+1)^2}\left(1 - 4^{-\alpha-1}(1+2(\alpha+1)\log 2)\right).$$

The case $\alpha = -1$ is easy and should be treated separately. □

3 Asymptotic distribution of $n\rho_n(x/n),\ 0 \le x \le n$.

For any n the distribution of $n\rho_n(x'/n),\ 0 \le x' \le n$, is $p_n(\tau/n)d\tau$ where $p_n(\cdot)$ is the density function of $\rho_n(x)$, $0 \le x \le 1$, introduced in Lemma 2.1. Therefore the study of the asymptotic distribution of $n\rho_n(x'/n),\ 0 \le x' \le n,\ n \to \infty$, is equivalent to the study of the $*$-weak convergence of the measure sequence $p_n(\tau/n)d\tau$. This study is the main purpose of Theorem 3.1 which also contains a statement concerning $*$-weak convergence of the sequence of measures $\tilde{\mu}_n = \mu_n(n,1)$ which assign the measure 1 to the numbers $np_{i,n}$ for $i = 1, \ldots, N(n)$:

$$\tilde{\mu}_n = \mu_n(n,1) = \sum_{i=1}^{N(n)} \delta(t - np_{i,n}),$$

where $p_{i,n}$ are defined in (7).

Theorem 3.1. *Let $\mathcal{F}_n$ be the Farey sequence of order n and $n \to \infty$. Then the measure sequence $\{\tilde{\mu}_n\}_n$ $*$-weakly converge to the measure*

$$\tilde{\mu} = 2\sum_{k=1}^{\infty} \varphi(k)\delta(t - \frac{1}{k}) \tag{23}$$

on $\mathcal{B}$ and the measure sequence $\{p_n(\tau/n)d\tau\}$ $*$-weakly converge to the measure $\tilde{p}(\tau)d\tau$ on $\mathcal{B}$ with the density (2). Moreover, for all $\tau \neq \frac{1}{2m}$, $m = 1, 2, \ldots$, the sequence $p_n(\tau/n)$ converge to $\tilde{p}(\tau)$ and for $n \to \infty$ and any $\delta > 1$

$$\int_0^\infty t^{\delta+1} d\tilde{\mu}_n(t) \to \int_0^\infty t^{\delta+1} d\tilde{\mu}(t) = 2\frac{\zeta(\delta)}{\zeta(\delta+1)} < \infty, \tag{24}$$

$$n^{\delta+1}\int_0^1 \rho_n^\delta(x)dx = \int_0^\infty \tau^\delta p_n(\tau/n)d\tau \to \int_0^\infty \tau^\delta \tilde{p}(\tau)d\tau = 2C_\delta \frac{\zeta(\delta)}{\zeta(\delta+1)} < \infty \tag{25}$$

where $C_\delta = 2^{-\delta}/(1+\delta)$ is as defined in (17) and the error terms in (24) and (25) have the order $O\left(n^{1-\delta}\right)$, $n \to \infty$.

Proof. In the course of the proof we shall use the notations of Section 2 and results of four lemmas, namely, Lemmas 2.1–2.4. The consideration of the measures $\tilde{\mu}_n = \mu_n(n, 1)$ and $p_n(\tau/n)d\tau$ means that we have put $F_n = n$ and $G_n = 1$ for the values of the normalization constants of Section 2. Certainly, we consider the Farey sequences as $\mathcal{F}_n$. (The corresponding partitions $\mathcal{P}_n$ of $[0, 1)$ will be called the Farey partitions.)

Lemma 2.1 implies that for every $n \geq 1$ the densities $p_n(\tau/n)$ and the measures $\tilde{\mu}_n$ are related via (14) and the application of Lemma 2.3 gives that for every $\delta > 1$ the moment of order δ of $\rho_n(x)$ can be represented through the Mellin transform, see (18), of the measure $\tilde{\mu}_n$:

$$\int_0^1 \rho_n^\delta(x)dx = \frac{1}{n}\int_0^n \rho_n^\delta(x'/n)dx' = \frac{1}{n^{\delta+1}}\int_0^\infty \tau^\delta p_n(\tau/n)d\tau = \frac{C_\delta}{n^{\delta+1}}M(\tilde{\mu}_n)(\delta+1)$$

where

$$M(\tilde{\mu}_n)(\delta+1) = \int_0^\infty t^{\delta+1}d\tilde{\mu}_n(t) = n^{\delta+1}\sum_{i=1}^{N(n)} p_{i,n}^{\delta+1} \,.$$

The Mellin transform of the measure $\tilde{\mu}$, defined via (23), is equal to

$$M(\tilde{\mu})(s) = 2\sum_{q=1}^{\infty} \frac{\varphi(q)}{q^s} = 2\frac{\zeta(s-1)}{\zeta(s)} < \infty \tag{26}$$

for all s such that $\operatorname{Re} s > 2$. (Here we have used the well-known relation between the Riemann ζ–function and the Euler φ–function, see [9], problem 29, ch.2)

Let us prove that for all $\delta > 1$

$$M(\tilde{\mu}_n)(\delta+1) \to M(\tilde{\mu})(\delta+1), \quad n \to \infty \,. \tag{27}$$

It is well known, see for example [10], that if p/q and p'/q' are two succesive terms in the Farey sequence $\mathcal{F}_n$ then

$$1 \leq q, q' \leq n, \; q \neq q', \; q + q' > n \,. \tag{28}$$

This implies that if the endpoints of the intervals $I_{i,n} \in \mathcal{P}_n$, that is, $x_{i-1,n}$ and $x_{i,n}$, have denominators q and q' and $q \leq q'$ then $q' > n/2$ and the length $p_{i,n} = x_{i,n} - x_{i-1,n} = 1/(qq')$ of $I_{i,n}$ can always be bounded as

$$\frac{1}{qn} \leq p_{i,n} \leq \frac{1}{q(n-q)} \,. \tag{29}$$

These bounds will be used for the intervals $I_{i,n}$ one of whose has a denominator $q \leq n/2$.

An upper bound for the length of the intervals $I_{i,n}$, when both endpoints have denominators $\geq n/2$, follows from the formula $p_{i,n} = 1/(qq')$:

$$\frac{1}{n^2} \leq p_{i,n} \leq \frac{4}{n^2} \,. \tag{30}$$

The bounds (29), (30) for $p_{i,n}$ give the following lower and upper bounds for $M(\tilde{\mu}_n)(\delta{+}1)$:

$$M(\tilde{\mu}_n)(\delta{+}1) \geq A_n = 2n^{\delta+1} \sum_{q=1}^{n/2} \varphi(q) \frac{1}{q^{\delta+1} n^{\delta+1}} = 2 \sum_{q=1}^{n/2} \frac{\varphi(q)}{q^{\delta+1}}, \tag{31}$$

$$M(\tilde{\mu}_n)(\delta{+}1) \leq B_n = 2n^{\delta+1} \left(\sum_{q=1}^{n/2} \varphi(q) \frac{1}{q^{\delta+1}(n-q)^{\delta+1}} + \frac{4^{\delta+1}}{n^{2(\delta+1)}} \sum_{q=n/2}^{n} \varphi(q) \right) \tag{32}$$

Since $\varphi(q) \leq q$ for all integers q,

$$B_n = 2n^{\delta+1} \sum_{q=1}^{n/2} \varphi(q) \frac{1}{q^{\delta+1}(n-q)^{\delta+1}} + O\left(\frac{1}{n^{\delta-1}}\right), \quad n \to \infty.$$

According to the finite difference formula for every $n \geq 1, 1 \leq q \leq n$ and $\delta > 1$

$$n^{\delta+1} - (n-q)^{\delta+1} \leq q(\delta+1)n^{\delta}$$

and therefore

$$0 \leq B_n - A_n = 2 \sum_{q=1}^{n/2} \varphi(q) \frac{n^{\delta+1} - (n-q)^{\delta+1}}{q^{\delta+1}(n-q)^{\delta+1}} + O\left(\frac{1}{n^{\delta-1}}\right) \leq$$

$$(\delta+1)\left(\frac{2}{n}\right)^{\delta+1} \sum_{q=1}^{n/2} \frac{1}{q^{\delta-1}} + O\left(\frac{1}{n^{\delta-1}}\right) = O\left(\frac{1}{n^{\delta-1}}\right)$$

when $n \to \infty$. Furthermore, using (26) we get for all $\delta > 1$:

$$0 \leq M(\tilde{\mu})(\delta{+}1) - A_n = 2 \sum_{q=n/2}^{\infty} \frac{\varphi(q)}{q^{\delta+1}} = O\left(\frac{1}{n^{\delta-1}}\right), \quad n \to \infty.$$

This implies (27). Applying now the first part of Lemma 2.4 we obtain the $*$-weak convergence of the sequence of measures $\tilde{\mu}_n$ to $\tilde{\mu}$ when $n \to \infty$. The statement of the theorem concerning the convergence of $p_n(\tau/n)$ to $\tilde{p}(\tau)$ follows from Lemma 2.2. The condition (13) obviously holds for $A = 2$ since $\tilde{\mu}_n([2,\infty)) = \tilde{\mu}([2,\infty)) = 0$ for any $n \geq 1$. The relation (25) follows from (24) and Lemma 2.3. □

4 Asymptotic distribution of $n^2\rho_n(x)$.

For any n the distribution of $n^2\rho_n(x)$, $0 \leq x \leq 1$, is $n^{-2}p_n(\tau/n^2)d\tau$, and we thus can consider the problem of studying the asymptotic distribution of $n^2\rho_n(x)$, $n \to \infty$, as the problem of the weak convergence of the sequence of probability measures $n^{-2}p_n(\tau/n^2)d\tau$.

Analogously with Theorem 3.1, in Theorem 4.1 one more associated measure sequence is also studied, this time this is the sequence of probability measures

$$\hat{\mu}_n = \mu_n(n^2, 1/N(n)) = \frac{1}{N(n)} \sum_{i=1}^{N(n)} \delta(t - n^2 p_{i,n})$$

which corresponds to the selection of F_n, G_n of Section 2 in the form $F_n = n^2$, $G_n = 1/N(n)$ where $N(n) = |\mathcal{F}_n| - 1 = \sum_{k=1}^{n} \varphi(k)$.

Theorem 4.1. *Let $\mathcal{F}_n$ be the Farey sequence of order n, and let the function $F(\cdot)$ and the constant M_α be defined via* (21) *and* (22)*, correspondingly. Then the sequence of probability measures $\hat{\mu}_n = \mu_n(n^2, 1/N(n))$ weakly converge, when $n \to \infty$, to the probability measure $\hat{\mu}$ on $\mathcal{B}$ with the cumulative distribution function $\hat{\mu}(\tau) = 1 - F(1/\tau)$, $\tau \geq 0$, the sequence of probability measures $n^{-2} p_n(\tau/n^2) d\tau$ in $\mathcal{M}$ weakly converge, when $n \to \infty$, to the probability measure $\hat{p}(\tau) d\tau$ in $\mathcal{M}$ with the probability density $\hat{p}(\tau) = \frac{6}{\pi^2} F(1/(2\tau))$, $\tau \geq 0$, and for all $\tau > 0$ the sequence $n^{-2} p_n(\tau/n^2)$ converge, when $n \to \infty$, to $\hat{p}(\tau)$. Moreover, for any $\delta < 2$ and $n \to \infty$*

$$\int_0^\infty \tau^\delta d\hat{\mu}_n(\tau) \to \int_0^\infty \tau^\delta d\hat{\mu}(\tau) = M_{-\delta} < \infty \,, \tag{33}$$

and for any $-1 < \delta < 1$ and $n \to \infty$

$$n^{2\delta} \int_0^1 \rho_n^\delta(x) dx = n^{-2} \int_0^\infty \tau^\delta p_n(\tau/n^2) d\tau \to \int_0^\infty \tau^\delta \hat{p}(\tau) d\tau = \frac{3}{(1+\delta)\pi^2 2^\delta} M_{-\delta-1} < \infty \tag{34}$$

To prove the theorem we need to introduce some notation and prove two more lemmas and Theorem 1.3.

Let p/q and p'/q' be neighbours in $\mathcal{F}_n$ such that $0 \leq p/q < p'/q' \leq 1$. The ordered pair (q, q') will go under the name of the neighbouring pair of denominators in $\mathcal{F}_n$.

Lemma 4.1. *The set of all neighbouring pairs of denominators in $\mathcal{F}_n$ coincides with the set of pairs of ordered integers*

$$\mathcal{Q}_n = \{(q, q') : \; q, q' \in \{1, 2, \ldots, n\}, \; (q, q') = 1, \; q + q' > n\} \,. \tag{35}$$

Proof. Let p/q and p'/q' be two neighbours in $\mathcal{F}_n$ such that $p/q < p'/q'$. Then the property of the Farey sequences (28) implies $(q, q') \in \mathcal{Q}_n$. Note that the number of different neighbouring pairs $(p/q, p'/q')$ in $\mathcal{F}_n$ equals $N(n) = \sum_{j=1}^{n} \varphi(j)$. The number of elements in $\mathcal{Q}_n$ also equals $N(n)$. Indeed, for a fixed $q \in \{1, \ldots, n\}$, the number of elements in the set

$$\mathcal{M}_{q,n} = \{q' : \; (q, q') = 1, \; q' \in \{n{-}q{+}1, \ldots, n\}\}$$

does not depend on n and equals $|\mathcal{M}_{q,n}| = \varphi(q)$, therefore

$$\mathcal{Q}_n = \bigcup_{q=1}^{n} \mathcal{M}_{q,n}, \quad |\mathcal{Q}_n| = \sum_{q=1}^{n} |\mathcal{M}_{q,n}| = \sum_{q=1}^{n} \varphi(q) = N(n)\,.$$

To $(q, q') \in \mathcal{Q}_n$, there can correspond at most one pair of neighbours $(p/q, p'/q')$ in $\mathcal{F}_n$: for such neighbours we have the equation $p'q - pq' = 1,\ 0 \le p < q,\ 1 \le p' \le q'$, and since $(q, q') = 1$, there is only one solution of this equation. Since, as pointed out, the number of elements in $\mathcal{Q}_n$ is equal to the number of neighbouring pairs in $\mathcal{F}_n$, the lemma follows. □

Lemma 4.2. *Consider the set of $\varphi(q)$ points on the unity circle $\mathbf{T}$*

$$Z_q = \{e^{2\pi i q'/q},\ \ q' = 1, \dots, q,\ \ (q', q) = 1\} \subseteq \mathbf{T}\,.$$

Then the sequence of Borel probability measures on $\mathbf{T}$

$$\lambda_q = \frac{1}{\varphi(q)} \sum_{\zeta \in Z_q} \delta_\zeta \tag{36}$$

converge, when $n \to \infty$, to the normalized Lebesque measure m on $\mathbf{T}$ and the convergence is uniform: for any arc $I_\alpha = \{e^{i\psi},\ \ -\alpha \le \psi \le \alpha\},\ \ 0 \le \alpha \le \pi$,

$$\lim_{q\to\infty} \lambda_q(I_\alpha e^{i\phi}) = \frac{\alpha}{\pi} = m(I_\alpha)$$

uniformly with respect to $\phi \in [0, 2\pi)$.

Proof. The fact of convergence of the measure sequence $\{\lambda_n\}_n$ to the uniform measure on $\mathbf{T}$ is equivalent to the asymptotic uniformity of the Farey sequence, the proof of this can be found, for example, in [11]. The fact that this convergence is uniform, follows from Lemma 2.5. □

Proof of Theorem 1.3. Define the trapezoid

$$\Delta = \Delta(\beta_1, \beta_2, \alpha_1, \alpha_2) = \{(x, y) \in [0, 1] \times [0, 1] : \beta_1 \le x < \beta_2, \alpha_1 \le \frac{1-y}{x} < \alpha_2\} \tag{37}$$

where $0 \le \alpha_1 < \alpha_2 \le 1$ and $0 \le \beta_1 < \beta_2 \le 1$.

The set of all trapezoids of the form (37) constitutes the set determining convergence, see [5], on the triangle $T = \{(x, y) :\ \ 0 \le x, y \le 1,\ \ x + y \ge 1\}$. To establish the weak convergence of the measure sequence $\{\nu_n\}_n$ to m, the uniform probability measure on T and thus the doubled Lebesgue measure on T, it is therefore sufficient to show that

$$\lim_{n\to\infty} \nu_n(\Delta) = m(\Delta) = (\alpha_2 - \alpha_1)(\beta_2^2 - \beta_1^2) \tag{38}$$

for all $0 \le \alpha_1 < \alpha_2 \le 1$, $0 < \beta_1 < \beta_2 \le 1$ and $\Delta = \Delta(\beta_1, \beta_2, \alpha_1, \alpha_2)$.

Let us fix $\alpha_1 < \alpha_2$, $\beta_1 < \beta_2$ and denote $n(q) = |\{q' : (\frac{q}{n}, \frac{q'}{n}) \in \Delta\}|$. For any q, $1 \le q \le n$, there exists $\gamma_q \in [0, 2\pi)$ such that

$$\frac{n(q)}{\varphi(q)} = \lambda_q(I_\alpha e^{i\gamma_q})$$

where $\alpha = \pi(\alpha_2 - \alpha_1)$ and λ_q is the measure (36). The statement of Lemma 4.2 implies that for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for all $n \geq n_0(\varepsilon)$ the inequality

$$\left| \frac{n(q)}{\varphi(q)} - (\alpha_2 - \alpha_1) \right| < \varepsilon$$

holds for all q such that $\beta_1 n \leq q \leq \beta_2 n$. Therefore for all $n \geq n_0(\varepsilon)$

$$\left| \nu_n(\Delta) - (\alpha_2 - \alpha_1) \sum_{q=\beta_1 n}^{\beta_2 n} \varphi(q)/N(n) \right| \leq$$

$$\frac{1}{N(n)} \sum_{q=\beta_1 n}^{\beta_2 n} \left| \frac{n(q)}{\varphi(q)} - (\alpha_2 - \alpha_1) \right| \varphi(q) \leq \varepsilon \frac{1}{N(n)} \sum_{q=\beta_1 n}^{\beta_2 n} \varphi(q) \leq \varepsilon \,.$$

The well–known summation formula for the Euler function

$$N(n) = \sum_{q=1}^{n} \varphi(q) = \frac{3}{\pi^2} n^2 + O(n \log n) \,, \quad n \to \infty \,, \tag{39}$$

implies that for all $0 < \beta_1 < \beta_2 \leq 1$

$$\sum_{q=\beta_1 n}^{\beta_2 n} \varphi(q) = \frac{3}{\pi^2} (\beta_1^2 - \beta_2^2) n^2 + O(n \log n) \,, \quad n \to \infty \,,$$

and therefore

$$\frac{1}{N(n)} \sum_{q=\beta_1 n}^{\beta_2 n} \varphi(q) \to \beta_1^2 - \beta_2^2 \quad \text{when} \quad n \to \infty \,.$$

We thus get (38), and this completes the proof. □

Proof of Theorem 4.1. Recall that the length of every interval $I_{i,n}$ in the Farey partition $\mathcal{P}_n$ equals $p_{i,n} = 1/(qq')$ where $(q, q') \in \mathcal{Q}_n$ is the ordered pair of the denominators of the endpoints of the interval, see Lemma 4.1. According to the definition of the measure ν_n, given in the introduction, for any $a > 0$

$$\hat{\mu}_n([a, \infty)) = \frac{1}{N(n)} \left| \{(q, q') \in \mathcal{Q}_n \,:\, \frac{n^2}{qq'} \geq a\} \right|$$

$$= \nu_n(\{(x, y) :\; 0 \leq x, y \leq 1, \; x + y \geq 1, \; xy \leq 1/a\})$$

Theorem 1.3 implies that the expression in the right-hand side of the last formula tends to $F(1/a) = \hat{\mu}([a, \infty))$, when $n \to \infty$, for any $a > 0$. For all $n \geq 1$ and $\tau > 0$ define

$$\hat{p}_n(\tau) = \frac{1}{N(n)} p_n(\tau/n^2) = 2\hat{\mu}_n([2\tau, \infty))$$

and note that for all $\tau > 0$

$$\frac{n^{-2}p_n(\tau/n^2)}{\hat{p}_n(\tau)} = \frac{N(n)}{n^2} = \frac{3}{\pi^2} + O(n^{-1}\log n), \quad n \to \infty .$$

Applying Lemma 2.2 we get that for all $\tau > 0$

$$\hat{p}_n(\tau) \to 2\hat{\mu}([2\tau, \infty)) = 2F(1/(2t)), \quad n \to \infty ,$$

and therefore

$$n^{-2}p_n(\tau/n^2) \to \hat{p}(\tau) = \frac{6}{\pi^2}F(1/(2t)), \quad n \to \infty ,$$

for all $\tau > 0$, where the explicit form of $\hat{p}(\tau)$ is given in (3). Lemma 2.2 also yields the weak convergence, when $n \to \infty$, of the probability measures $\hat{p}_n(\tau)d\tau \in \mathcal{M}$ to the limiting measure $\hat{p}(\tau)d\tau$.

We are going now to apply the second part of Lemma 2.4 to prove (33). To do this, we have to verify the conditions (19) and (20). Since the measures $\hat{\mu}_n$ and $\hat{\mu}$ are the probability measures, (20) obviously holds for $\alpha = 0$. To demonstrate the validity of (19), it is enough to show that for any $a < 2$

$$\sup_{n \geq 1} \int_0^\infty t^a d\hat{\mu}_n(t) < \infty . \tag{40}$$

If $a < 0$ then the left–hand side of (30) gives

$$\int_0^\infty t^a d\hat{\mu}_n(t) = \frac{n^{2a}}{N(n)} \sum_{i=1}^{N(n)} p_{i,n}^a \leq \frac{n^{2a}}{N(n)} \sum_{i=1}^{N(n)} n^{-2a} = 1$$

Assume now that $0 < a < 2$. Then analogously to (32), with $\delta + 1 = a$, we get

$$\int_0^\infty t^a d\hat{\mu}_n(t) = M(\tilde{\mu}_n)(a) = \frac{n^{2a}}{N(n)} \sum_{i=1}^{N(n)} p_{i,n}^a$$

$$\leq 2\frac{n^{2a}}{N(n)} \left(\sum_{q=1}^{n/2} \varphi(q) \frac{1}{q^a(n-q)^a} + \frac{4^a}{n^{2a}} \sum_{q=n/2}^{n} \varphi(q) \right) .$$

Since $\varphi(q) \leq q$, $N(n) \geq n(n+1)/4$ and

$$\sum_{q=1}^{n/2} q^{1-a} \leq 1 + \int_1^n x^{1-a} dx \leq n^{2-a}/(2-a) + 1 ,$$

for all integers n and $0 < a < 2$, we get

$$\int_0^\infty t^a d\hat{\mu}_n(t) \leq 2\frac{n^a}{N(n)} \sum_{q=1}^{n/2} q^{1-a} \left(\frac{n}{n-q} \right)^a + 2\frac{4^a}{N(n)} \sum_{q=1}^{n} q \leq$$

$$8n^{a-2}2^a \sum_{q=1}^{n/2} q^{1-a} + 4^{a+1} \le 2^{a+3}(1 + 1/(2-a)) + 4^{a+1} .$$

We thus have shown the validity of (40) and therefore completed the justification of (33). The validity of (34) follows now from Lemmas 2.2, 2.3 and the relation $\hat{p}(\tau) = \frac{6}{\pi^2}\hat{\mu}([2\tau, \infty))$. □

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