# Estimating the covariance function of isotopic fields on the sphere

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Outline

- covariance function, examples
- estimation, distribution
- cosmic variance
- simulation
- nonlinear problems

- cosmic microwave background (CMB)
- $\mathbb{S}_2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$
- a real-valued random field $T(\omega, x) = T(x)$, $\omega \in \Omega, x \in \mathbb{S}_2$, with $\mathbb{E}\,T(x) = 0$.
- isotropic, if $\mathbb{E}\,T(x)^2 < \infty$, and $\mathbb{E}\,T(x)\,T(y) = \mathbb{E}\,T(gx)\,T(gy)$ for any $g \in SO(3)$, $x, y \in \mathbb{S}_2$, where $SO(3)$ rotational group

Colatitude $\vartheta \in [0, \pi]$ and longitude $\varphi \in [0, 2\pi]$.
Orthogonal system complex-valued spherical harmonics $Y_\ell^m$, where $\ell = 0, 1, 2, \ldots$, and $m = -\ell, -\ell + 1, \ldots - 1, 0, 1, \ldots, \ell - 1, \ell$.
$T(x)$ is mean square continuous,

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(x), \qquad (1)$$

Coefficients given by

$$a_{\ell m} = \int_{\mathbb{S}_2} T(x)\, Y_\ell^{m*}(x)\, \Omega(dx), \tag{2}$$

$\Omega(dx) = \sin\vartheta d\vartheta d\varphi$ is the Lebesgue measure of surface area on $\mathbb{S}_2$

$$\mathbb{E}a_{\ell m} = 0,\ \mathbb{E}a_{\ell m}a_{kn}^{*} = f_\ell \delta_{\ell,k}\delta_{m,n}. \tag{3}$$

In particular $a_{\ell m}$ is normal with mean 0 and variance $\mathbb{E}\left|a_{\ell m}\right|^2 = f_\ell$.

$T(x)$ is Gaussian.

$$\mathcal{C}_2(x_1, x_2) = \mathcal{C}(\cos\gamma) = \sum_{\ell=0}^{\infty} f_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\gamma), \tag{4}$$

where $P_\ell$ denotes the Legendre polynomial
angular spectrum $f_\ell$

$$f_\ell = 2\pi \int_0^\pi \mathcal{C}(\cos\gamma) P_\ell(\cos\gamma) \sin\gamma d\gamma. \tag{5}$$

**Laplace-Beltrami model** on $\mathbb{S}_2$.

$$\left(\triangle_B - c^2\right) T_B = \partial W_B,$$

$\partial W_B$ denotes the white noise, The Laplace-Beltrami operator is

$$\triangle_B = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}.$$

A direct calculation leads to the spectrum

$$f_\ell = \frac{1}{\left(\ell\left(\ell+1\right)+c^2\right)^2}, \tag{6}$$

for $T_B$, and the covariance function

$$\mathcal{C}\left(\cos\gamma\right) = \frac{\sigma^2}{4\pi}\sum_{\ell=0}^{\infty}\frac{2\ell+1}{\left(\ell\left(\ell+1\right)+c^2\right)^2}P_\ell\left(\cos\gamma\right),$$

# Laplacian

Homogeneous and isotropic random field $T$ on $\mathbb{R}^d$

$$\left(\triangle - c^2\right)^{\nu} T = \partial W,$$

$\triangle = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$, Laplace operator and $\partial W$ white noise

$$\mathcal{C}\left(\cos\gamma\right) \equiv \left(\frac{2\sin\left(\gamma/2\right)}{|c|}\right)^{2\nu-\frac{d}{2}} K_{2\nu-\frac{d}{2}}\left(2\left|c\right|\sin\left(\gamma/2\right)\right), \tag{7}$$

where $K_{\nu}$ is the modified Bessel

$$f_{\ell} = 2\pi\int_0^{\pi}\mathcal{C}\left(\cos\gamma\right)P_{\ell}\left(\cos\gamma\right)\sin\gamma d\gamma.$$

# Laplacian

For $\nu = 1$, $d = 3$, we have

$$\mathcal{C}\left(\cos\gamma\right) = \frac{1}{\left(2\pi\right)^{3/2}}\sqrt{\frac{\sin\left(\gamma/2\right)}{2\left|c\right|}}K_{1/2}\left(2\left|c\right|\sin\left(\gamma/2\right)\right).$$

with spectrum

$$f_\ell = \int_0^\infty J^2_{\ell+1/2}\left(\lambda\right)\frac{\lambda}{\left(\lambda^2+c^2\right)^2}d\lambda.$$

$$\mathcal{C}\left(\cos\gamma\right) = \frac{1}{8\pi\left|c\right|}e^{-2\left|c\right|\sin\left(\gamma/2\right)}.$$

$$f_\ell = \frac{1}{4\left|c\right|}\int_0^\pi e^{-2\left|c\right|\sin\left(\gamma/2\right)}P_\ell\left(\cos\gamma\right)\sin\gamma d\gamma.$$

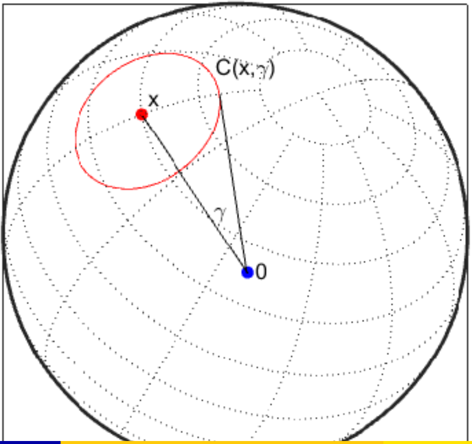
$$\mathcal{C}\left(\cos\gamma\right)=\frac{1}{8\pi\left|c\right|}e^{-2\left|c\right|\sin(\gamma/2)}$$

The von Mises Fisher density on the sphere

$$f\left(\gamma;\underline{\mu},\kappa\right)=\frac{\sqrt{\kappa}}{\left(2\pi\right)^{3/2}I_{1/2}\left(\kappa\right)}e^{-\kappa\cos(\gamma/2+\pi/2)} \qquad (8)$$

The covariance function $\mathcal{C}_2\left(x_1\cdot x_2\right)$ in (4) is strictly positive definite if all $f_\ell$ are $\geq 0$, and only finitely many of them are zero .
If the coefficients of the series expansion of $f\left(\gamma;\underline{\mu},\kappa\right)$ are positive then the same function may also serve as a covariance function. .

$$\widehat{\mathcal{C}}\left(\cos\gamma\right)=\int_{\mathbb{S}_2}\int_{C(x,\gamma)}T\left(x\right)T\left(y_\gamma\left(x,\psi\right)\right)\frac{d\psi}{2\pi}\frac{\Omega\left(dx\right)}{4\pi}.$$



## Theorem

*$T(x)$ is Gaussian then*

$$\widehat{\mathcal{C}}(\cos\gamma) = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\left(\left|\widehat{a}_{\ell 0}\right|^2 + 2\sum_{m=1}^{\ell}\left|\widehat{a}_{\ell m}\right|^2\right) P_\ell(\cos\gamma), \tag{9}$$

*Characteristic function*

$$\varphi(z) = \prod_{\ell=0}^{\infty}\frac{1}{\left(1 - \mathrm{i}zf_\ell P_\ell(\cos\gamma)/2\pi\right)^{\ell+1/2}}. \tag{10}$$

Let $\gamma_m \in [0, \pi]$, $m = 1, 2, \ldots j$, be given angles, then the joint characteristic function of $\widehat{\mathcal{C}}(\cos\gamma_1), \widehat{\mathcal{C}}(\cos\gamma_2), \ldots, \widehat{\mathcal{C}}(\cos\gamma_j)$ is

$$\varphi(z_1, z_2, \ldots z_j) = \prod_{\ell=0}^{\infty} \frac{1}{\left(1 - \mathrm{i} f_\ell \left(\sum_{m=1}^{j} z_m P_\ell(\cos\gamma_m)\right)/2\pi\right)^{\ell+1/2}}.$$

The error

$$R = \widehat{\mathcal{C}}\left(\cos\gamma\right) - \mathcal{C}\left(\cos\gamma\right). \tag{11}$$

$$R \stackrel{d}{=} \frac{1}{4\pi}\sum_{\ell=0}^{\infty}\left(2\ell+1\right) f_\ell P_\ell\left(\cos\gamma\right)\left(\frac{U_{2\ell+1}}{2\ell+1}-1\right). \tag{12}$$

$U_{2\ell+1}/\left(2\ell+1\right)$ is Gamma distributed

**Theorem**

*The characteristic function of $R$ is*

$$\varphi\left(z\right) = \exp\left(\sum_{k=2}^{\infty}\left(\frac{\mathrm{i}z}{2\pi}\right)^k \frac{1}{2k}\sum_{\ell=0}^{\infty}\left(2\ell+1\right)\left(f_\ell P_\ell\left(\cos\gamma\right)\right)^k\right) \tag{13}$$

*which is a Rosenblatt type characteristic function.*

It is *infinitely divisible* and *selfdecomposable*

$$\varphi(z) = \exp\left(\int_0^\infty [e^{\mathrm{i}zx} - 1 - \mathrm{i}zx]\nu(x)\,dx\right),$$

with Lévy density

$$\nu(x) = \frac{1}{2x}\sum_{\ell=0}^{\infty}(2\ell+1)\exp\left(-\frac{x}{8\pi f_\ell P_\ell(\cos\gamma)}\right).$$

$\varphi(z)$ belongs to the Thorin class $T(\mathbb{R})$, with Thorin measure, $a_\ell = 8\pi f_\ell P_\ell(\cos\gamma)$

$$U(dx) = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1)\delta_{1/a_\ell}(x),$$

# Cosmic Variance

Consider a sample path of the field

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_\ell^m(x).$$

$f_\ell = \mathbb{E}\widehat{f}_\ell = \mathbb{E}\left|\widehat{a}_{\ell m}\right|^2$. By introducing

$$\widehat{f}_\ell = \frac{1}{2\ell+1}\left(\sum_{m=-\ell}^{\ell} \left|\widehat{a}_{\ell m}\right|^2\right),$$

which has the property $\mathbb{E}\widehat{f}_\ell = f_\ell$,and

$$\mathrm{Var}\left(\widehat{f}_\ell\right) = \frac{2f_\ell^2}{2\ell+1},$$

## Definition

the *cosmic variance* as

$$\mathbb{E}\left(\frac{f_\ell - \widehat{f}_\ell}{f_\ell}\right)^2 = \frac{2}{2\ell+1}, \tag{14}$$

Consider then the remainder

$$R_M = \frac{1}{4\pi} \sum_{\ell=M}^{\infty} (2\ell+1) f_\ell P_\ell(\cos\gamma) \left( \frac{U_{2\ell+1}}{2\ell+1} - 1 \right),$$

where the sum starts at $\ell = M$. Since $R_M$ is associated to the sample path $T_M(x)$ defined as

$$T_M(x) = T(x) - \sum_{\ell=0}^{M-1} \sum_{m=-\ell}^{\ell} \widehat{a}_{\ell m} Y_\ell^m(x), \tag{15}$$

and since $\widehat{a}_{\ell m}$ are good approximations of the current values of the dynamics, see (2), (not like the estimation of $f_\ell$), $T_M(x)$ is a good estimator of the reminder

$$\mathcal{C}_M(\cos\gamma) = \mathcal{C}(\cos\gamma) - \frac{1}{4\pi} \sum_{\ell=0}^{M-1} f_\ell (2\ell+1) P_\ell(\cos\gamma).$$

## Theorem

*$R_M/\sigma_M$ is* **asymptotically standard normal**

$$\sigma_M^2 = \frac{2}{(4\pi)^2}\sum_{\ell=M}^{\infty}(2\ell+1)\, f_\ell^2 P_\ell^2(\cos\gamma),$$

*Berry–Esseen bound*

$$\sup_{x\in\mathbb{R}}\left|P\left(R_M \le x\right) - \Phi(x)\right| \le 0.7056\kappa_{3,M},$$

*where $\Phi$ is the standard normal CDF, and $\kappa_{3,M}$ the third cumulant*

$$\kappa_{3,M} = \frac{1}{(2\pi)^3\sigma_M^3}\sum_{\ell=M}^{\infty}(2\ell+1)\, f_\ell^3 P_\ell^3(\cos\gamma).$$

$$\widehat{\mathcal{C}}\left(\cos\gamma\right) = \int_{\mathbb{S}_2}\int_{C(x,\gamma)} T\left(x\right) T\left(x_\gamma\left(\varphi, x\right)\right) \frac{d\varphi}{2\pi}\frac{\Omega\left(dx\right)}{4\pi},$$

is discretized, as follows

$$\int_{C(x,\gamma)} T\left(x\right) T\left(x_\gamma\left(\varphi, x\right)\right) \frac{d\varphi}{2\pi} \tag{16}$$

$$\sim \frac{1}{n_x}\left(T\left(x\right) - \overline{T}\right) \sum_{x\cdot x_j = \cos\gamma} \left(T\left(x_j\right) - \overline{T}\right), \tag{17}$$

where $x_j$ denote locations of pixel centers, $\overline{T} = \left(1/N_{pix}\right)\sum_x T\left(x\right)$, $n_x$ is pairs of $x$ and $x_j$, such that $x \cdot x_j = \cos\gamma$.

$$
\begin{aligned}
&\widehat{\mathcal{C}}\left(\cos\gamma\right) \\
=\ &\frac{1}{N_{pix}}\sum_{i}\frac{1}{n_{x_i}}\sum_{j;x_i\cdot x_j=\cos\gamma}\left(T\left(x_i\right)-\overline{T}\right)\left(T\left(x_j\right)-\overline{T}\right),
\end{aligned}
$$

$$f_\ell = 2\pi \int_0^\pi \mathcal{C}_{(L)}(\cos\gamma)\, P_\ell(\cos\gamma) \sin\gamma d\gamma$$
$$= 2\pi \int_{-1}^{1} \mathcal{C}_{(L)}(y)\, P_\ell(y)\, dy.$$

Exact value, Gauss-Legendre quadrature,

$$f_\ell = 2\pi \sum_{i=1}^{L+1} w_i \mathcal{C}_{(L)}(y_i)\, P_\ell(y_i), \tag{18}$$

nodes $y_1, \ldots, y_{L+1}$ are the roots of the Legendre polynomial $P_{L+1}(x)$, while $w_1, \ldots, w_{L+1}$ are the corresponding weights, exact for polynomials up to order $2L+1$,

# Laplace- Beltrami model

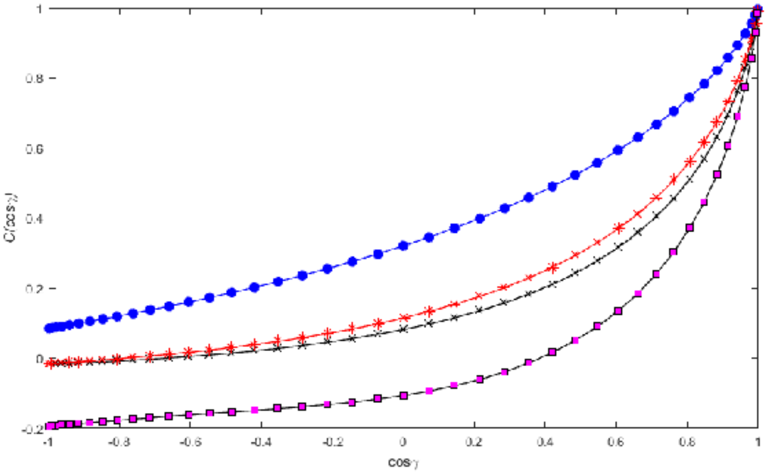



Figure: Correlation functions from top to bottom: (a) with $\widehat{c}$ ; (b)

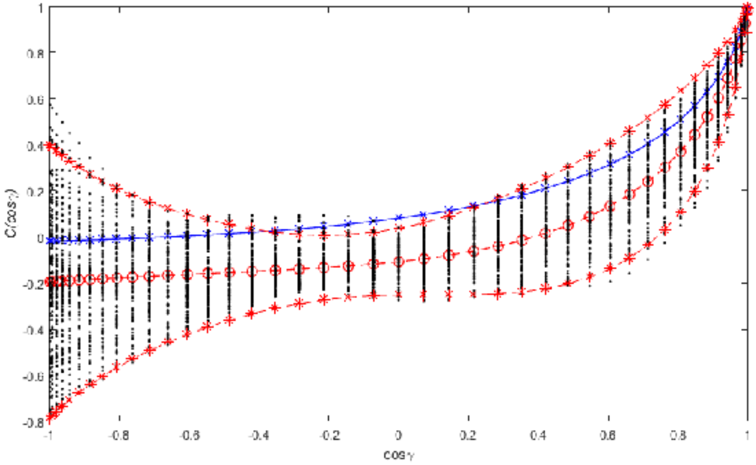



Figure: Theoretical $c = 2$; confidence curve, estimated, lower confidence curve. Dots are individual simulations from 1 to 100.

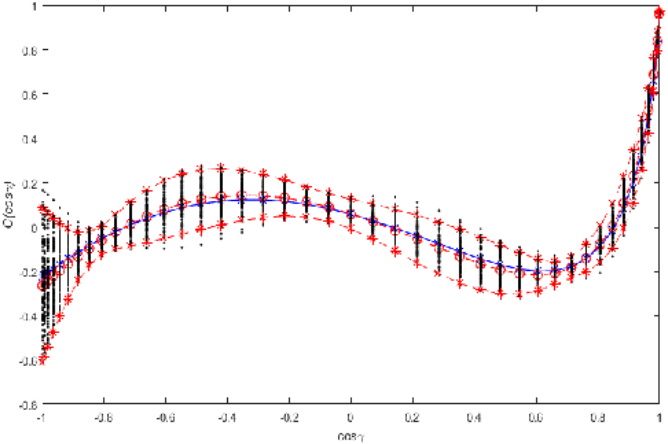



Figure: Estimation of the correlation function when $f_\ell = 0$, $\ell = 0 : 3$.The estimated is closed to the theoretical. Dots are the individual simulations from 1 to 100.

# Non-Gaussian, Flat-sky

$$X\left(\underline{x}\right)=\int_{\mathbb{R}^2}e^{i\underline{x}\cdot\underline{\omega}}Z\left(d\underline{\omega}\right),\quad \underline{\omega},\underline{x}\in\mathbb{R}^2,$$

with a finite spectral measure

$$E\left|Z\left(d\underline{\omega}\right)\right|^2=F_0\left(d\underline{\omega}\right).$$

In term of polar coordinates

$$X(r,\varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) \tag{19}$$

where $J_\ell$ the Bessel function , $\underline{x} = (r,\varphi)$, $\underline{\omega} = (\rho,\eta)$ are polar coordinates

$$Z_\ell(\rho d\rho) = \int_0^{2\pi} i^\ell e^{-i\ell\eta}\ Z(\rho d\rho d\eta).$$

$$X(r,\varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) A(\rho) W_\ell(\rho d\rho),$$

## Definition

A homogenous stochastic field $X\left(\underline{x}\right)$ is strictly isotropic if all finite dimensional distributions of $X\left(\underline{x}\right)$ are invariant under all rotations $g \in SO\left(2\right)$, i.e. all finite dimensional distributions of $X\left(\underline{x}\right)$ and $\Lambda\left(g\right) X\left(\underline{x}\right)$ are the same.

## Example

Consider a Gaussian homogenous and isotropic random field $X\left(\underline{x}\right)$, then $X\left(\underline{x}\right) + X^2\left(\underline{x}\right)$ is clearly a homogenous and isotropic non-Gaussian random field.

## Isotropy assumption

$$\begin{aligned} &\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right)\right) \\ = \;& e^{i(\ell_1+\ell_2)\gamma}\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right)\right), \end{aligned}$$

for each $\gamma$, hence either $\ell_1 + \ell_2 = 0$, or $\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right)\right) = 0$.

$$\begin{aligned} &\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), \dots, Z_{\ell_p}\left(\rho_p d\rho_p\right)\right) \\ = \;& e^{i(\ell_1+\ell_2\cdots+\ell_p)\gamma}\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), \dots, Z_{\ell_p}\left(\rho_p d\rho_p\right)\right), \end{aligned}$$

that is either $\ell_1 + \ell_2 \cdots + \ell_p = 0$, or $\operatorname{Cum}\left(Z_{\ell_1}\left(\rho_1 d\rho_1\right), Z_{\ell_2}\left(\rho_2 d\rho_2\right), \dots, Z_{\ell_p}\left(\rho_p d\rho_p\right)\right) = 0$, Gaussian random fields that

$$\mathrm{Cov}\left(X\left(\underline{x}\right),X\left(\underline{y}\right)\right)=\int_0^\infty J_0\left(\rho r\right)F\left(\rho d\rho\right),$$

where $r=\left|\underline{x}-\underline{y}\right|$, Absolutely continuos spectral measure we have $F\left(\rho d\rho\right)=\sigma^2\left|A\left(\rho\right)\right|^2\rho d\rho$ and therefore

$$\mathcal{C}_2\left(r\right)=\int_0^\infty J_0\left(\rho r\right)\sigma^2\left|A\left(\rho\right)\right|^2\rho d\rho,$$

$$\sigma^2\left|A\left(\rho\right)\right|^2=\frac{1}{2\pi}\int_0^\infty J_0\left(\rho r\right)\mathcal{C}_2\left(r\right)rdr,$$

.

homogenous and isotropic stochastic field $X\left(\underline{x}\right)$

$$\begin{aligned}&\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right)\right)\\ =\ &\operatorname{Cum}\left(X\left(0\right), X\left(g\left(\underline{x}_2-\underline{x}_1\right)\right), X\left(\left|\underline{x}_3-\underline{x}_1\right|\underline{n}\right),\right)\end{aligned}$$

The bicovariance $\operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2\right), X\left(r_3\underline{n}\right),\right)$ depends on the lengths $r_2$, $r_3=\left|\underline{x}_3\right|$,and the angle $\varphi$ between them,

$$\mathcal{C}_3\left(r_1, r_2, r_3\right)=\operatorname{Cum}\left(X\left(0\right), X\left(\underline{x}_2\right), X\left(r_3\underline{n}\right)\right)$$

$$\begin{aligned}&\mathcal{C}_3\left(\varphi, r_2, r_3\right)\\ =\ &2\iint_0^\infty\int_0^\pi \mathcal{T}_3\left(\alpha,\rho_2,\rho_3\right|\varphi, r_2, r_3\left.\right) S_3\left(\alpha,\rho_2,\rho_3\right) d\alpha\prod_{k=2}^3\rho_k\end{aligned}$$

$S_3(\alpha, \rho_2, \rho_3)$ (complex values?) depends on wave numbers $(\rho_1, \rho_2, \rho_3)$ such that $\rho_1, \rho_2, \rho_3$ should form a triangle.

$$\mathcal{T}_3\left(\alpha, \rho_2, \rho_3 \middle| \varphi, r_2, r_3\right) = \sum_{\ell=-\infty}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\alpha),$$

$$S_3(\rho_1, \rho_2, \rho_3) = \frac{1}{2\pi} \iint_0^\infty \int_0^\pi \mathcal{T}_3\left(\alpha, \rho_2, \rho_3 \middle| \varphi, r_2, r_3\right) \mathcal{C}_3(r_1, r_2, r_3)\, d\varphi \prod_{k=2}^{3} r$$

bispectrum of a homogenous and isotropic stochastic field is **real valued**.

# Homogenous isotropic field with Hermite rank 2

Gaussian-field in $\mathbb{R}^3$

$$X\left(\underline{x}\right) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m\left(\widehat{\underline{x}}\right) \int_0^\infty j_\ell\left(\rho r\right) a\left(\rho\right) W_\ell^m\left(\rho^2 d\rho\right),$$

where $W\left(d\underline{\omega}\right)$ is Gaussian, $\mathrm{E}\left|W(d\underline{\omega})\right|^2 = \frac{1}{(2\pi)^3} d\underline{\omega}$. A non-Gaussian model

$$H\left(\underline{x}\right) = X\left(\underline{x}\right) + f_{NL}\left(X^2\left(\underline{x}\right) - EX^2\left(\underline{x}\right)\right)$$

The coefficient $f_{NL}$ is measuring the nonlinearity of the CMB observations for instance. $X^2\left(\underline{x}\right) - EX^2\left(\underline{x}\right)$ is Hermite polynomial of degree 2 of the Gaussian random

$H(\underline{x})$ is an elementary, very simple case of a chaotic field

$$H(\underline{x}) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^{3\times k}} \exp(it\Sigma\underline{\omega}_{1:k}) f_k(\underline{\omega}_{1:k}) W(d\underline{\omega}_{1:k}) . \tag{20}$$

where $\underline{\omega}_{1:k} = [\underline{\omega}_1, \underline{\omega}_2, \ldots, \underline{\omega}_k]$, $W(d\underline{\omega}_{1:k})$ is the multiple Wiener-Itô stochastic spectral measure, The nonlinear model we shall consider is

$$H(\underline{x}) = X_1(\underline{x}) + X_2(\underline{x})$$

where the quadratic transfer function $a_2(\rho_{1:2})$ is symmetric function of its variables.

$$X_1(\underline{x}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r)\, a_1(\rho)\, W_\ell^m(\rho^2 d\rho) \tag{21}$$

and

$$\begin{aligned} X_2(\underline{x}) &= (4\pi)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \\ &\times \sum_{\ell_{1:2}=0}^{\infty} C_{\ell_1;\ell_2} \iint_0^\infty \prod_{q=1}^{2} j_{\ell_q}(\rho_q r)\, a_2(\rho_{1:2})\, W_{\ell_1,\ell_2}^{\ell,m} \end{aligned}$$

Now the spectrum of $H(\underline{x})$ is simple, it is the sum of spectra, since – because of the orthogonality of multiple Wiener-Itô integrals – we have

$$\mathcal{C}_2(r) = \frac{1}{2\pi^2}\int_0^\infty j_0(\rho r)\left(|a_1(\rho)|^2 + S_{X_2}(\rho)\right)\rho^2 d\rho,$$

therefore the spectrum is $\left(|a_1(\rho)|^2 + S_{X_2}(\rho)\right)/2\pi^2$.

$$\mathrm{Cum}\left(H\left(r_1\widehat{\underline{x}}\right), H\left(r_2 N\right), H\left(\underline{0}\right)\right)$$

$$= 6\left(4\pi\right)\iint_0^\infty a_1\left(\rho_1\right) a_1\left(\rho_2\right) a_2\left(\rho_{1:2}\right)\prod_{q=1}^{2} j_0\left(\rho_q r\right)\prod_{q=1}^{2}\rho_q^2 d\rho_q$$

$$+ 8\left(4\pi\right)^4\sum_{\ell}\left(2\ell+1\right) P_\ell\left(\cos\vartheta\right)$$

$$\times\iiint_0^\infty j_0\left(\rho_1 r_1\right) j_\ell\left(\rho_2 r_1\right) j_0\left(\rho_2 r_2\right) j_\ell\left(\rho_3 r_2\right) a_2\left(\rho_{1:2}\right) a_2\left(\rho_2, \rho_3\right.$$