Spectral expansions of random sections of spin and tensor bundles

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The main ideas

1. A cosmological random field $\mathbf{X}(x)$ that takes values in a finite-dimensional Hilbert space E over any of number fields $\mathbb{R}$ or $\mathbb{C}$, generates a representation of the symmetry group G of the field $\mathbf{X}(x)$ in some Hilbert space $H_{\mathbf{X}}$ of E-valued random vectors (or tensors). Moreover, the inner product in $H_{\mathbf{X}}$ is G-invariant.
2. The above representation is equivalent to a representation acting in an invariant subspace, say H, of a special representation of G that is called an induced representation $L^2(G/K; E)$.
3. Conversely, for any invariant subspace H of $L^2(G/K; E)$, we can construct an invariant random section $\mathbf{X}(x)$ of the vector bundle $(G \times_K E, p, G/K)$ that generates a representation equivalent to H.

A plan of the presentation

1. Define an induced representation and its invariant random sections.
2. Determine the structure of the induced representation.
3. Construct a correspondence between invariant random fields and invariant subspaces of the induced representation.
4. Give examples from astrophysics.

Notation

1. G — a compact Lie group.
2. K — its closed subgroup.
3. G/K — the quotient space of G under the equivalence relation $g \sim gk$ for $k \in K$.
4. E — a finite-dimensional representation of K over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
5. $G \times_K E$ — the quotient space of $G \times E$ under the equivalence relation $(g, \mathbf{v}) \sim (gk, k^{-1} \cdot \mathbf{v})$ for $k \in K$.
6. μ — a G-invariant measure on Borel sets of G/K.

Vector bundles

Observe that the projection $G \times E \to G$, $(g, \mathbf{v}) \mapsto g$ respects the equivalence relations and induces the map $p\colon G \times_K E \to G/K$. The triple $(G \times_K E, p, G/K)$ is a vector bundle, with the total space $G \times_K E$, the base space G/K, the projection p and the fibers $p^{-1}(\{x\})$ isomorphic to E for every $x \in G/K$, that is, every $x \in G/K$ has a local trivialisation.

Definition

A local trivialisation near a point $x \in G/K$ is an open neighbourhood U of x, and a homeomorphism $\varphi\colon U \times E \to p^{-1}(U)$ such that for all $y \in U$,

- $(p \circ \varphi)(y, \mathbf{v}) = y$ for all $\mathbf{v} \in E$,
- the map $\mathbf{v} \mapsto \varphi(y, \mathbf{v})$ is a linear isomorphism between the linear spaces E and $p^{-1}(\{y\})$.

Induced representations

A section of the bundle $(G \times_K E, p, G/K)$ is a map $\mathbf{s}\colon G/K \to G \times_K E$ with $p \circ \mathbf{s} = \mathrm{id}$. Let $L^2(G/K; E)$ be the Hilbert space of all square-integrable sections with norm

$$\|\mathbf{s}\|^2 = \int_{G/K} \|\mathbf{s}(x)\|_E^2 \, \mathrm{d}\mu(x).$$

Definition

The induced representation acts by $(g \cdot \mathbf{s})(x) = g \cdot \mathbf{s}(g^{-1}x)$.

Random fields

Definition

A random field in the vector bundle $(G \times_K E, p, G/K)$ is a set of random vectors $\{\, \mathbf{X}(x) \colon x \in G/K \,\}$ such that $\mathbf{X}(x)$ is a $p^{-1}(\{x\})$-valued random vector.

Definition

A random field $\mathbf{X}(x)$ is called G-invariant if for all positive integers n, for all n-tuples of distinct points $x_1, \dots, x_n \in G/K$ and for all $g \in G$ the random matrices $(\mathbf{X}(x_1), \dots, \mathbf{X}(x_n))$ and $(\mathbf{X}(g^{-1}x_1), \dots, \mathbf{X}(g^{-1}x_n))$ have the same distribution.

Definition

A random field $\mathbf{X}(x)$ is called second-order if $\mathsf{E}[\|\mathbf{X}(x)\|_E^2] < \infty$ for all $x \in G/K$.

A link to representations

Let $H_{\mathbf{X}}$ be the closure of the linear hull of the random vectors $\{\, \mathbf{X}(x) \colon x \in G/K \,\}$ in the Hilbert space $L^2(\Omega; E)$ of the second-order E-valued random vectors with inner product

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{L^2(\Omega;E)} = \mathsf{E}[\langle \mathbf{X}, \mathbf{Y} \rangle_E].$$

Consider the map $\mathbf{X}(x) \mapsto \mathbf{X}(g^{-1}x)$ and extend it by linearity to the linear hull of the random vectors $\{\, \mathbf{X}(x) \colon x \in G/K \,\}$.

Definition

A random field $\mathbf{X}(x)$ is called mean-square continuous if the above map can be extended by continuity to $H_{\mathbf{X}}$.

Observe: the extended map is a continuous representation of G in $H_{\mathbf{X}}$. If $\mathbf{X}(x)$ is G-invariant, then the inner product in $H_{\mathbf{X}}$ is G-invariant.

More definitions from representation theory

Definition

An intertwining operator f between representations V and W is a linear map that satisfies $f(g \cdot \mathbf{v}) = g \cdot f(\mathbf{v})$.

Definition

The representations V and W are called equivalent if the linear space $\mathrm{Hom}_G(V, W)$ of all intertwining operators contains an invertible operator.

Definition

A subspace $U \subset V$ is called invariant if $g \cdot \mathbf{u} \in U$ for $g \in G$ and $\mathbf{u} \in U$.

Definition

A representation $V \neq \{0\}$ is called irreducible if it has no invariant subspaces other than $\{\mathbf{0}\}$ and V.

The main result

Theorem

If a random field $\mathbf{X}(x)$ is second-order and G-invariant, then there is an invariant subspace $H \subset L^2(G/K; E)$ and an invertible intertwining operator between $H_{\mathbf{X}}$ and H.

Schur's lemma and its consequences

Lemma (Schur)

Let V and W be irreducible representations of a group G. An intertwining operator $f\colon V \to W$ is either zero or an isomorphism.

Let E be an irreducible representation of a compact Lie group G, necessarily finite-dimensional. By Schur's lemma, the linear space $D(E) = \mathrm{Hom}_G(E, E)$ is a division algebra over $\mathbb{K}$. Frobenius' theorem states that $D(E) = \mathbb{C}$ when $\mathbb{K} = \mathbb{C}$ and $D(E) \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ when $\mathbb{K} = \mathbb{R}$, where $\mathbb{H}$ is the skew field of quaternions.

The evaluation $D(E) \times E \to E$, $(f, \mathbf{v}) \mapsto f(\mathbf{v})$ turns E into left $D(E)$-linear space.

For an arbitrary representation V, the composition $\mathrm{Hom}_G(E, V) \times D(E) \to \mathrm{Hom}_G(E, V)$, $(f, \varphi) \mapsto f \circ \varphi$ turns $\mathrm{Hom}_G(E, V)$ into right $D(E)$-linear space.

An algebraic decomposition of a locally finite representation

For a representation V, let V_s be its locally finite part, that is, a subspace of V generated by finite-dimensional continuous invariant subspaces.

By [Bröcker and tom Dieck (1995), Chapter 3, Proposition 1.7], the evaluation map

$$c_E\colon \operatorname{Hom}_G(E, V) \otimes_{D(E)} E \to V, \qquad f \otimes \mathbf{v} \to f(\mathbf{v})$$

is an intertwining operator between $\operatorname{Hom}_G(E, V) \otimes_{D(E)} E$ and its image $c(E) = c_E(\operatorname{Hom}_G(E, V) \otimes_{D(E)} E)$. Moreover, $c(E)$ is contained in V_s, and the algebraic direct sum of the spaces $c(E)$ over all equivalence classes of irreducible representations of G is equal to V_s.

Definition

The subspace $c(E)$ is called the E-isotypical part of V_s. The number $n(E, c(E)) = \dim_{\mathbb{K}} \operatorname{Hom}_G(E, c(E))$ is called the multiplicity of E in V_s.

Computing multiplicities

For a general representation, it's a hard work. For an induced representation $L^2(G/K;E)$, it's easy. Let V be a representation of G, and let $\operatorname{res}^G_K V$ be its restriction to K.

Theorem (Frobenius reciprocity)

The linear spaces $\operatorname{Hom}_G(V, L^2(G/K;E))$ *and* $\operatorname{Hom}_K(\operatorname{res}^G_K V, E)$ *are isomorphic.*

We have $\dim_{\mathbb{K}} \operatorname{Hom}_G(V, L^2(G/K;E)) \otimes_{D(V)} V = \dim_{\mathbb{K}} c(V)$, or

$$\dim_{D(V)} \operatorname{Hom}_G(V, L^2(G/K;E)) \dim_{D(V)} V \dim_{\mathbb{K}} D(V) = n(V, c(V)) \dim_{\mathbb{K}} V.$$

Then

$$n(V, c(V)) = \frac{\dim_{\mathbb{K}} \operatorname{Hom}_K(\operatorname{res}^G_K V, E)}{\dim_{\mathbb{K}} D(V)}.$$

Computing multiplicities, cont.

When E is irreducible, we have $\dim_{\mathbb{K}} \operatorname{Hom}_K(\operatorname{res}^G_K V, E) = n(E, \operatorname{res}^G_K V)$, and

$$n(V, c(V)) = \frac{n(E, \operatorname{res}^G_K V)}{\dim_{\mathbb{K}} D(V)}.$$

Otherwise assume that E contains n_1 copies of the irreducible representation E_1, ..., n_p copies of the irreducible representation E_p. Then we have $\dim_{\mathbb{K}} \operatorname{Hom}_K(\operatorname{res}^G_K V, E) = n_1 n(E_1, \operatorname{res}^G_K V) + \cdots + n_p n(E_p, \operatorname{res}^G_K V)$ and

$$n(V, c(V)) = \frac{1}{\dim_{\mathbb{K}} D(V)} \sum_{i=1}^{p} n_i n(E_i, \operatorname{res}^G_K V).$$

Notation

Suppose that E is the direct sum of the irreducible representations $E_1, \dots, E_N$. Denote the components of an invariant random field $\mathbf{X}(x)$ by $X_j^{(i)}(x)$, $1 \leq i \leq N$, $1 \leq j \leq \dim_{\mathbb{K}} E_i$. Let P_i be the orthogonal projection from E to E_i. Let H_s^i be the representation induced by E_i.
Denote by $\hat{G}_K(E_i)$ the set of all $V \in \hat{G}$ with $n(E_i, \operatorname{res}_K^G V) > 0$ and by V_0 the trivial representation of the group G: $V_0(g) = 1$ for all $g \in G$. For each $V \in \hat{G}_K(E_i)$, fix an orthonormal basis ${}_{E_i}\mathbf{Y}_{Vn}^m(x)$ in the space of the nth copy of the representation V, where $1 \leq m \leq \dim_{\mathbb{K}} V$, $1 \leq n \leq n(V, c_i(V))$.

Spectral expansion

Theorem

The random field $\mathbf{X}(x)$ *has the form*

$$X_j^{(i)}(x) = \sum_{V\in\hat{G}_K(E_i)} \sum_{m=1}^{\dim_{\mathbb{K}} V} \sum_{n=1}^{n(V,H_s^i)} \sum_{k=1}^{\dim_{\mathbb{K}} E_i} a_{Vnmik} ({}_{E_i}Y_{Vn}^m)_j(x),$$

where

$$a_{Vnmik} = \int_{G/K} X_k^{(i)}(x) \overline{({}_{E_i}Y_{Vn}^m)_k(x)} \, \mathrm{d}\mu(x).$$

If $V \neq V_0$*, then* $\mathsf{E}[a_{Vnmik}] = 0$*. Finally,* $\mathsf{E}[a_{Vnmik}\overline{a_{V'n'm'i'k'}}] = \delta_{VV'}\delta_{mm'}R_{ik,i'k'}^{Vn}$ *with*

$$\sum_{i=1}^{N} \sum_{V\in\hat{G}_K(E_i)} \sum_{n=1}^{n(V,H_s^i)} \dim_{\mathbb{K}} V \operatorname{tr}[P_i R^{Vn} P_i] < \infty.$$

Spin bundles

Put $\mathbb{K} = \mathbb{C}$, $G = \mathrm{SO}(3)$, $K = \mathrm{SO}(2)$, $E_s = \mathbb{C}^1$ with the action of K given by $\varphi \mapsto \mathrm{e}^{-\mathrm{i}s\varphi}$, $s \in \mathbb{Z}$. The restriction of the irreducible unitary representation V^ℓ to K contains one copy of the representation E_s if and only if $\ell \in \{|s|, |s|+1, \dots\}$. By Frobenius reciprocity,

$$H_s = \sum_{\ell=|s|}^{\infty} \oplus V^\ell.$$

The basis vectors of the subspace of the Hilbert space $L^2(S^2, \mathrm{SO}(3) \times_{\mathrm{SO}(2)} E_s)$ where the representation W_ℓ acts, are called spin s spherical harmonics and are denoted by ${}_sY_{\ell m}(\mathbf{n})$. In particular, ${}_0Y_{\ell m}(\mathbf{n}) = Y_{\ell m}(\mathbf{n})$.

Spin bundles, cont.

Let $(Q+\mathrm{i}U)(\theta,\varphi)$ be an isotropic random field in the *spin bundle* $(\mathrm{SO}(3)\times_{\mathrm{SO}(2)} H_s, p, S^2)$ with $s=2$. Then it has the form

$$(Q\pm\mathrm{i}U)(\mathbf{n})=\sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}^{(\pm 2)}\,{}_{\pm 2}Y_{\ell m}(\mathbf{n}),$$

where

$$a_{\pm 2,\ell m}=\int_{\mathbb{C}P^1}(Q\pm\mathrm{i}U)(\mathbf{n})\,\overline{{}_{\pm 2}Y_{\ell m}(\mathbf{n})}\,\mathrm{d}\mathbf{n}.$$

The real-valued random fields $Q(\mathbf{n})$ and $U(\mathbf{n})$ describe the Stokes parameters of the linear polarisation of the Cosmic Microwave Background. This expansion appeared in [Zaldarriaga and Seljak (1997)].

The representations of $\mathrm{O}(2)$

Put $\mathbb{K} = \mathbb{R}$, $G = \mathrm{O}(3)$, $K = \mathrm{O}(2)$. Let E be the real linear space of Hermitian linear operators over a two-dimensional complex linear space with inner product $(A, B) = \operatorname{tr} AB$, where the action of K is given by $k \cdot A = kAk^{-1}$, $k \in K$, $A \in E$.

The group $\mathrm{O}(2)$ has the following irreducible orthogonal representations: $E^{0+}(k) = 1$, $E^{0-}(k) = \det k$, and the representations E^m, $m \geq 1$, acting by

$$E^m \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(m\varphi) & -\sin(m\varphi) \\ \sin(m\varphi) & \cos(m\varphi) \end{pmatrix},$$

$$E^m \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(m\varphi) & \sin(m\varphi) \\ \sin(m\varphi) & -\cos(m\varphi) \end{pmatrix}.$$

The structure of the representation E

This time the representation E is reducible and contains three inequivalent irreducible components. The first component is E^{0+}. It acts in the one-dimensional subspace E_+ generated by the matrix

$$\frac{1}{\sqrt{2}}\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The elements of this space are scalars.

The second component is E^{0-}. It acts in the one-dimensional subspace E_- generated by the matrix

$$\frac{1}{\sqrt{2}}\sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

The elements of this subspace are pseudo-scalars.

The structure of the representation E, cont.

The third irreducible component is E^2. It acts in the two-dimensional space E_2 of symmetric trace-free matrices generated by

$$\frac{1}{\sqrt{2}}\sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices σ_i are known as *Pauli matrices*.

The group $\mathrm{O}(3)$ is the Cartesian product of its subgroups $\mathrm{SO}(3)$ and $Z_2^c = \{\pm e\}$. Therefore, $V = V^{\ell\pm} = V^{\ell} \otimes V^{\pm}$, where V^{ℓ} is the irreducible orthogonal representation of the group $\mathrm{SO}(3)$, V^{+} is the trivial representation of the group Z_2^c, and V^{-} is the representation $g \mapsto \det g$ of Z_2^c.

A lemma

Lemma

The restriction of the representation $V^{\ell\pm}$ to the subgroup $\mathrm{O}(2)$ has the form

$$\begin{aligned} \operatorname{res}^G_K V^{2\ell\pm} &= E^{0\pm} \oplus E^1 \oplus \cdots \oplus E^{2\ell}, \\ \operatorname{res}^G_K V^{(2\ell+1)\pm} &= E^{0\mp} \oplus E^1 \oplus \cdots \oplus E^{2\ell+1}. \end{aligned}$$

We have $E = E_+ \oplus E_- \oplus E_2$. It follows that

$$\mathrm{O}(3) \times_{\mathrm{O}(2)} E = (\mathrm{O}(3) \times_{\mathrm{O}(2)} E_+) \oplus (\mathrm{O}(3) \times_{\mathrm{O}(2)} E_-) \oplus (\mathrm{O}(3) \times_{\mathrm{O}(2)} E_2),$$

and similar equality is true for the spaces of the square-integrable sections of the above bundles: $H = H_+ \oplus H_- \oplus H_2$. By Lemma, the representation E^{0+} belongs to the restrictions to $\mathrm{O}(2)$ of the representations V^{0+}, V^{1-}, V^{2+}, …. By Frobenius reciprocity,

$$n(V^{\ell(-1)^\ell}, H_+) = \frac{n(E^{0+}, \operatorname{res}^G_K V^{\ell(-1)^\ell})}{\dim_{\mathbb{R}} D(V^{\ell(-1)^\ell})}.$$

How to calculate $\dim_{\mathbb{R}} D(V)$ for a real irreducible representation V?

Define the *extended representation* by $e^{\mathbb{C}}_{\mathbb{R}} V = \mathbb{C} \otimes_{\mathbb{R}} V$. By [Bröcker and tom Dieck (1995), Chapter 2, Proposition 6.6, Theorem 6.7], there may be three cases:

1. $e^{\mathbb{C}}_{\mathbb{R}} V$ is irreducible. In this case, $D(V) = \mathbb{R}$ and $\dim_{\mathbb{R}} D(V) = 1$.
2. $e^{\mathbb{C}}_{\mathbb{R}} V$ is a direct sum of two non-equivalent components. In this case, $D(V) = \mathbb{C}$ and $\dim_{\mathbb{R}} D(V) = 2$.
3. $e^{\mathbb{C}}_{\mathbb{R}} V$ is a direct sum of two equivalent components. In this case, $D(V) = \mathbb{H}$ and $\dim_{\mathbb{R}} D(V) = 4$.

It is easy to check that all the real irreducible representations of G belong to the first class. We have

$$H_+ = \sum_{\ell=0}^{\infty} \oplus V^{\ell(-1)^{\ell}}, \qquad H_- = \sum_{\ell=0}^{\infty} \oplus V^{\ell(-1)^{\ell+1}}$$

and

$$H_2 = \sum_{\ell=2}^{\infty} \oplus (V^{\ell+} \oplus V^{\ell-}).$$

The intensity of the CMB

An isotropic random field in the bundle $(\mathrm{O}(3) \times_{\mathrm{O}(2)} E_+, p, S^2)$ takes the form

$$I(\mathbf{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a^{l}_{\ell m} Y_{\ell m}(\mathbf{n}),$$

and describes the first Stokes parameter, the intensity $I(\mathbf{n})$ of the Cosmic Microwave Background, which is proportional to the fourth power of its absolute temperature, $T(\mathbf{n})$, by the Stephan–Boltzmann law. Like $E(\mathbf{n})$, this is a scalar field, a section of the vector bundle generated by the trivial representation of the subgroup $\mathrm{O}(2)$.

The circular polarisation of the CMB

For the bundle $(\mathrm{O}(3) \times_{\mathrm{O}(2)} E_-, p, S^2)$ we have

$$V(\mathbf{n}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a^V_{\ell m} Y_{\ell m}(\mathbf{n}).$$

This field describes the circular polarisation $V(\mathbf{n})$ of the Cosmic Microwave Background. Like $B(\mathbf{n})$, this is a pseudo-scalar field, an isotropic random section of the vector bundle generated by the representation $g \mapsto \det g$ of the subgroup $\mathrm{O}(2)$.

The linear polarisation of the CMB

An isotropic random section of the bundle $(\mathrm{O}(3)\times_{\mathrm{O}(2)} E_2, p, S^2)$ has the form

$$\begin{pmatrix} Q(\mathbf{n}) & U(\mathbf{n}) \\ U(\mathbf{n}) & -Q(\mathbf{n}) \end{pmatrix} = \sum_{\ell=2}^{\infty}\sum_{m=-\ell}^{\ell}[a^G_{\ell m}Y^G_{\ell m}(\mathbf{n}) + a^C_{\ell m}Y^C_{\ell m}(\mathbf{n})],$$

where we denote by $\{\, Y^G_{\ell m}(\mathbf{n})\colon \ell \geq 2, -\ell \leq m \leq \ell \,\}$ a basis in the space $V^{\ell(-1)^{\ell}}$, while $\{\, Y^C_{\ell m}(\mathbf{n})\colon \ell \geq 2, -\ell \leq m \leq \ell \,\}$ is a basis in the space $V^{\ell(-1)^{\ell+1}}$. Comparing our expansion with [Kamionkowski et al (1997), Equation 2.10], we identify the introduced basis functions with tensor spherical harmonics described there.

The overdensity field and the Newtonian potential

Let $\delta(r, \mathbf{n})$ be the overdensity field:

$$\delta(r, \mathbf{n}) = \frac{\rho(r, \mathbf{n}) - \bar{\rho}}{\bar{\rho}},$$

where $\rho(r, \mathbf{n})$ is the matter density at a point $(r, \mathbf{n})$, and $\bar{\rho}$ is the average matter density. The overdensity field is not observable because most part of matter is dark and invisible. To overcome this difficulty, we proceed in three steps.

1. Relate the overdensity field $\delta(r, \mathbf{n})$ and the Newtonian potential $\Phi(r, \mathbf{n})$ by Poisson's equation

$$\nabla^2 \Phi(r, \mathbf{n}) = \frac{3\Omega_M H_0^2}{2a(r)} \delta(r, \mathbf{n}),$$

where Ω_M is the dimensionless matter density, H_0 is the current value of the Hubble parameter, and $a(r)$ is the dimensionless scale factor.

The lensing potential

2. Define the lensing potential by

$$\phi(r, \mathbf{n}) = \frac{2}{c^2} \int_0^r \Phi(r', \mathbf{n}) \frac{r - r'}{rr'} \, \mathrm{d}r',$$

where c is the speed of light in a vacuum.

Spin rising operators

3. For a fixed $s \in \mathbb{Z}$, define a differential operator $\eth$ (in fact, a family of operators) by

$$\eth = s\cot\theta - \frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}.$$

Then we have

$$\eth_s Y_{\ell m}(\mathbf{n}) = \begin{cases} 0, & \text{if } \ell = s \geq 0, \\ \sqrt{(\ell - s)(\ell + s + 1)}_{s+1} Y_{\ell m}(\mathbf{n}), & \text{otherwise.} \end{cases}$$

In other words, $\eth$ maps a section of the spin-s bundle to a section of spin-$s+1$ bundle, hence the name spin-rising operator.

Spin lowering operators

3. The conjugate operator

$$\eth^* = s\cot\theta - \frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}$$

lowers the spin:

$$\eth^*{}_sY_{\ell m}(\mathbf{n}) = \begin{cases} 0, & \text{if } \ell = -s \geq 0, \\ \sqrt{(\ell+s)(\ell-s+1)}\,{}_{s-1}Y_{\ell m}(\mathbf{n}), & \text{otherwise,} \end{cases}$$

hence the name spin-lowering operator

The distortion fields

Define the distortion fields ${}_sX((r,\mathbf{n})$ by

$$\begin{aligned}
{}_0\kappa(r,\mathbf{n}) &= \frac{1}{4}(\eth\eth^* + \eth^*\eth)\phi(r,\mathbf{n}),\\
{}_1\mathscr{F}(r,\mathbf{n}) &= -\frac{1}{6}(\eth^*\eth\eth + \eth\eth^*\eth + \eth\eth\eth^*)\phi(r,\mathbf{n}),\\
{}_2\gamma(r,\mathbf{n}) &= \frac{1}{2}\eth^2\phi(r,\mathbf{n}),\\
{}_3\mathscr{G}(r,\mathbf{n}) &= -\frac{1}{2}\eth^3\phi(r,\mathbf{n}).
\end{aligned}$$

The constructed fields are called the magnification, the first flexion, the shear, and the third flexion. In contrast to the overdensity field, they are observable. To find their spectral expansions, use the idea formulated by M.Ĭ. Yadrenko in [Yadrenko (1963)].

The (uncomplete) spectral expansion of the distortion fields

Suppose these fields are mean-square continuous. The restriction of the field ${}_sX(r,\mathbf{n})$ to the centred sphere of a fixed radius $r_0 > 0$ is an isotropic random section of the bundle $(\mathrm{O}(3)\times_{\mathrm{O}(2)} E_s, p, S^2)$ and has the form

$$ {}_sX(r_0,\mathbf{n}) = \sum_{\ell=s}^{\infty}\sum_{m=-\ell}^{\ell} {}_sa_{\ell m}(r_0)\,{}_sY_{\ell m}(\mathbf{n}), \qquad 0\le s\le 3. $$

Varying r_0, we obtain

$$ {}_sX(r,\mathbf{n}) = \sum_{\ell=s}^{\infty}\sum_{m=-\ell}^{\ell} {}_sa_{\ell m}(r)\,{}_sY_{\ell m}(\mathbf{n}), $$

where ${}_sa_{\ell m}(r)$ is a sequence of stochastic processes satisfying $\mathsf{E}[{}_sa_{\ell m}(r)] = 0$ unless $s=\ell=m=0$. Moreover, we have

$$ \mathsf{E}[{}_sa_{\ell m}(r_1)\overline{{}_sa_{\ell' m'}(r_2)}] = \delta_{\ell\ell'}\delta_{mm'}\,{}_sC_\ell(r_1,r_2), $$

with

$$ \sum^{\infty}(2\ell+1)\,{}_sC_\ell(r,r) < \infty, \qquad r\ge 0. $$

Parceval frames

Assume the following: the distortion field is Gaussian, has a.s. continuous sample paths and is observed in the ball of radius R. Then, the stochastic processes ${}_sa_{\ell m}(r)$ are Gaussian, independent, a.s. continuous, and their distributions depend on ℓ but do not depend on m. Each process generates a Gaussian centred measure ${}_s\mu_\ell$ in the real separable Banach space $C[0,R]$. Let ${}_sH_\ell$ be the reproducing kernel Hilbert space of ${}_s\mu_\ell$ with inner product $\langle\cdot,\cdot\rangle$, see [Vakhania et al (1987)].

A sequence $\{\, {}_sf_{\ell j}(r) \colon j \geq 1 \,\}$ of the elements of ${}_sH_\ell$ is called a Parceval frame [Luschgy and Pagès (2009)] if

$$\sum_{j=1}^{\infty}\langle {}_sf_{\ell j}, h\rangle\, {}_sf_{\ell j} = h$$

for all $h \in {}_sH_\ell$, where the series converges in the norm of ${}_sH_\ell$. By adding zeroes, finite sequences may also serve as frames.

Expansions of components

By [Luschgy and Pagès (2009), Theorem 1] the process ${}_s a_{\ell m}(r)$ has a.s. uniformly convergent expansion

$$ {}_s a_{\ell m}(r) = \sum_{j=1}^{\infty} {}_s f_{\ell j}(r)\, {}_s X_{\ell m j} $$

if and only if the sequence $\{\, {}_s f_{\ell j}(r) \colon j \geq 1 \,\}$ is a Parceval frame for ${}_s H_\ell$. The spectral expansion of the distortion field takes the form

$$ {}_s X(r, \mathbf{n}) = \sum_{\ell=s}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{j=1}^{\infty} {}_s f_{\ell j}(r)\, {}_s X_{\ell m j}\, {}_s Y_{\ell m}(\mathbf{n}). $$

In other words, each "component"

$$ {}_s X_\ell(r, \mathbf{n}) = \sum_{m=-\ell}^{\ell} {}_s a_{\ell m}(r)\, {}_s Y_{\ell m}(\mathbf{n}) $$

of the distortion random field ${}_s X(r, \mathbf{n})$ is expanded with respect to a *separable* Parceval frame

$$ \{\, {}_s f_{\ell j}(r)\, {}_s Y_{\ell m}(\mathbf{n}) \colon j \geq 1, -\ell \leq m \leq \ell \,\}. $$

The complete spectral expansion of the distortion fields

For each $\ell \geq s$, one can choose a suitable basis $\{\, {}_s f_{\ell j}(r) \colon j \geq 1 \,\}$ in the Hilbert space $L^2([0, R], r^2 \, \mathrm{d}r)$. These may be wavelets, orthogonal polynomials, etc. Assume that there are real numbers ${}_s c_{\ell j}$ such that the sequence $\{\, {}_s c_{\ell j} \, {}_s f_{\ell j}(r) \colon j \geq 1 \,\}$ is a Parceval frame for ${}_s H_\ell$. The distortion random field takes the form

$$ {}_s X(r, \mathbf{n}) = \sum_{\ell=s}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{j=1}^{\infty} {}_s c_{\ell j} \, {}_s f_{\ell j}(r) \, {}_s X_{\ell m j} \, {}_s Y_{\ell m}(\mathbf{n}). $$

The obtained expansion may serve as a base for the further statistical analysis of the distortion random field.

For further reading

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