On the choice of a linear reccurent formula for the SSA forecast

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Introduction: SSA algorithm

- Time series $(f_0, \ldots, f_{N-1})$, $f_j = h_j + \varepsilon_j$, where $(h_0, \ldots, h_{N-1})$ is a signal of rank r and ε_j is a noise
- Parameter: window length L, $K = N + 1 - L$, $r \le L < K$
- Trajectory matrix

$$\mathbf{X} = \begin{pmatrix} f_0 & f_1 & \cdots & f_{K-1} \\ f_1 & f_2 & \cdots & f_K \\ \vdots & & & \vdots \\ f_{L-1} & f_L & \cdots & f_{N-1} \end{pmatrix}$$

- SVD

$$\mathbf{X}\mathbf{X}^T = \sum_{m=1}^{L} \sqrt{\lambda_m} U_m U_m^T, \qquad \mathbf{X} = \sum_{m=1}^{L} \sqrt{\lambda_m} U_m V_m^T$$

- Reconstructed series (RS) is orthogonal projection (diagonal averaging) of $\sum_{m=1}^{r} \sqrt{\lambda_m} U_m V_m^T$
- Forecast using LRF and RS

Linear Reccurent Formula (LRF)

LRF of r-order

$$h_j = a_1 h_{j-1} + \ldots + a_r h_{j-r}$$

Characteristic polynomial of the LRF

$$P(t) = t^r - a_r - a_{r-1}t - \ldots - a_1 t^{r-1}$$

- From SSA theory, there is a unique minimal LRF (it is of order equal to the rank of time series) and many LRFs of greater order.

Example.
Suppose $h_j = \sum_{m=1}^{r} c_m e^{2\pi\lambda_m j}$, $\lambda_m \neq \lambda_s \in \mathbb{C}$.
Then $P(t) = \prod_{m=1}^{r}(t - \lambda_m)$ for minimal LRF.

Calculation of LRF

- SSA LRF has order L and $P(t) = p^T(1, t, \ldots, t^L)$ where

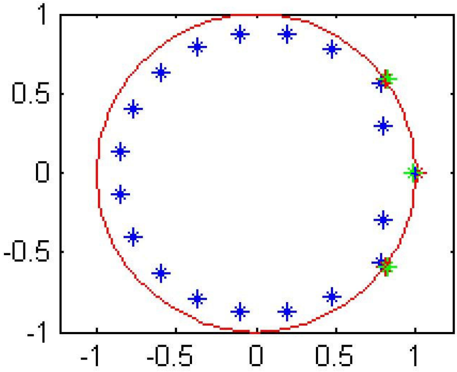
$$p = U_{r+1,L} U_{r+1,L}^T e_L^T$$

$$U_{r+1,L} = (U_{r+1} \vdots \ldots \vdots U_L), \ e_L = (0, \ldots, 0, 1)^T \in \mathbb{R}^L$$

- LRF of ESPRIT has order r and $P(t)$ equal to a characteristic polynomial of the matrix $\overline{U_{1,r}}^+ \underline{U_{1,r}} \in \mathbb{R}^{r \times r}$ where

$U_{1,r} = (U_1 \vdots \ldots \vdots U_r)$

Plot of roots of characteristic polynomial of LRF



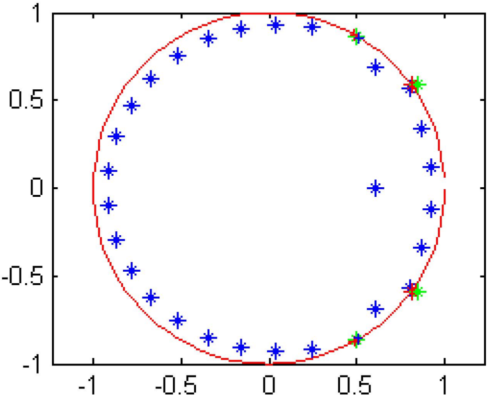
$h_j = 3 \cdot 1.01^j + 0.5 \sin(2\pi j/10)$, $r = 3$, $N = 100$, $L = 20$

red — minimal true LRF

green — ESPRIT LRF

blue — SSA LRF

Plot of roots of characteristic polynomial of LRF



$h_j = \sin(2\pi j/6) + 0.5\sin(2\pi j/10)$, $r = 4$, $N = 100$, $L = 30$

red — minimal true LRF

green — ESPRIT LRF

blue — SSA LRF

Why e_L in SSA LRF?

$$P(t) = p^T(1, t, \dots, t^L)$$

$$p = U_{r+1,L} U_{r+1,L}^T e_L^T$$

Consider a general form

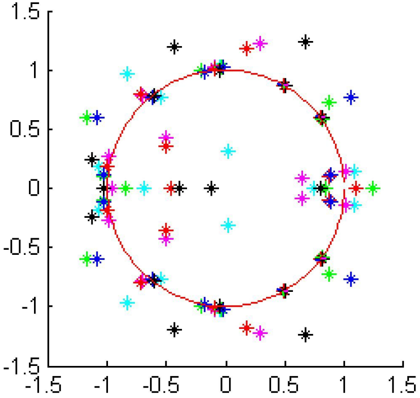
$$p = U_{r+1,L} U_{r+1,L}^T q^T, \quad q \in \mathbb{R}^L$$

Rewrite in a form

$$p = U_{r+1,L} w = \sum_{j=1}^{L-r} w_j U_{j+r}, \;\; w = (w_1, \dots, w_{L-r}) = U_{r+1,L}^T q^T \in \mathbb{R}^r$$

Plot of roots of $P(t)$ for different LRF

$P(t) = p^T(1, t, \ldots, t^L)$, $p = U_{r+1,L}w$



$h_j = \sin(2\pi j/6) + 0.5\sin(2\pi j/10)$, $r = 4$, $N = 100$, $L = 20$

red — $p = U_{r+1}$, $w = e_1$

green — $p = U_{r+2}$, $w = e_2$

blue — $p = U_{r+3}$, $w = e_3$, magenta, cyan, black

Why e_L in SSA LRF?

$P(t) = p^T(1, t, \ldots, t^L)$

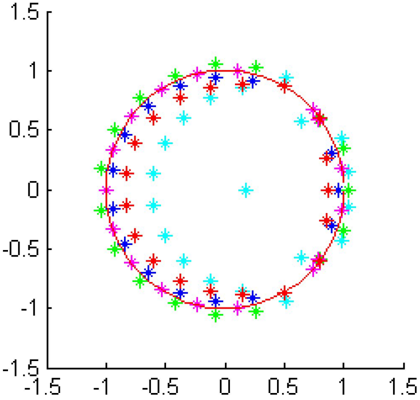
$$p = U_{r+1,L}U_{r+1,L}^T q^T$$

p is a projection of q to subspace generated by columns of $U_{r+1,L}$
$P(t)$ is a projection of $Q(t) = q^T(1, t, \ldots, t^L)$ to some subspace of polynomials.

How roots of $P(t)$ and $Q(t)$ are connected?

Plot of roots of $P(t)$ for different $Q(t)$

$$P(t) = p^T(1, t, \ldots, t^L), \quad p = U_{r+1,L}U_{r+1,L}^T q^T$$



$h_j = \sin(2\pi j/6) + 0.5\sin(2\pi j/10)$, $r = 4$, $N = 100$, $L = 20$

red — $Q(t) = t^L$,

magenta — $Q(t) = t^L + 1$

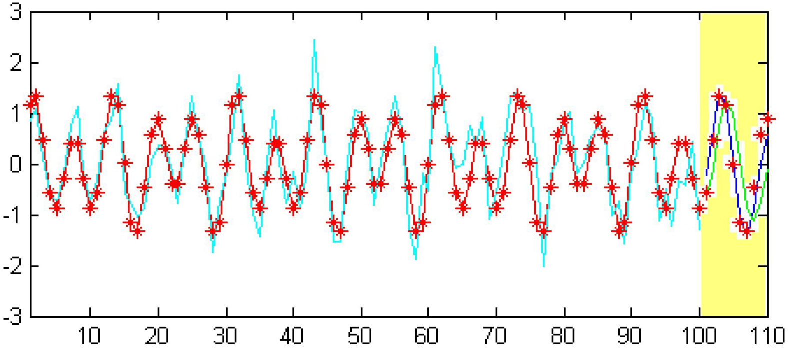
blue — $Q(t) = t^L - 0.95^L$,

green — $Q(t) = t^L - 1.05^L$,

cyan — $Q(t) = (t - 0.2)^L$

Plot of signal and its forecast

$$p = U_{r+1,L} U_{r+1,L}^T e_L^T$$



$h_j = \sin(2\pi j/6) + 0.5\sin(2\pi j/10)$, $r = 4$, $N = 100$, $L = 30$

red — true time series $(h_1, \ldots, h_N)$

cyan — time series with noise $(f_1, \ldots, f_N)$

green — forecast by ESPRIT LRF

blue — forecast by SSA LRF

Different LRF for investigation

- F1 true minimal LRF for RS
- F2 minimal LRF estimated via ESPRIT for RS
- F3 SSA LRF for RS
- F4 SSA LRF with replacement the phases of the main roots by true phases for RS
- F5 SSA LRF for true series
- F6 true SSA LRF for RS

Questions

- relation between F1 (true minimal LRF), F2 (ESPRIT LRF) and F3 (SSA LRF)
- relation between F3 and F4. What happen if we rewrite the phases of the main root by true phases in SSA LRF
- relation between F3 and F4+F5. How SSA LRF error splitts to the error related to the coefficients of LRF and the error from initial conditions for LRF
- dependence on L

Error is $\sqrt{\sum_{i=N+1}^{N+10}(h_i-\hat{h}_i)^2/10}$ where $\hat{h}_i$ is a forecast value of h_i.

Models

$$f_j = h_j + \varepsilon_j \quad , j = 1, \ldots, N, \quad N = 100$$

- $h_j = 1.1 \sin(2\pi j/10)$, $r = 2$
- $h_j = 3 \cdot 1.01^j + 0.5 \sin(2\pi j/10)$, $r = 3$
- $h_j = \sin(2\pi j/17) + 0.5 \sin(2\pi j/10)$, $r = 4$
- $h_j = \sin(2\pi j/6) + 0.5 \sin(2\pi j/10)$, $r = 4$

$L = 20, 30, 40, 50$

ε_j are i.i.d. $\mathrm{N}(0, \sigma^2)$, $\sigma = 0.5, 0.1$

Average forecast errors, $h_j = 1.1\sin(2\pi j/10)$, $r = 2$, $\sigma = 0.5$

L	F1	F2	F3	F4	F5	F6
20	0.165	0.179	0.161	0.149	0.043	0.133
30	0.141	0.154	0.148	0.131	0.052	0.113
40	0.136	0.150	0.152	0.130	0.061	0.106
50	0.144	0.159	0.158	0.134	0.079	0.102

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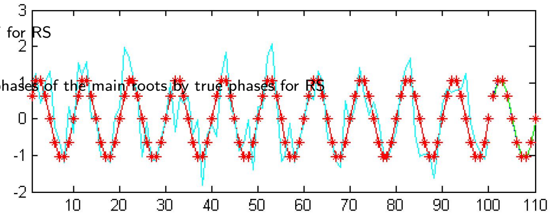

Average forecast errors, $h_j = 3 \cdot 1.01^j + 0.5\sin(2\pi j/10)$, $r = 3$, $\sigma = 0.5$

L	F1	F2	F3	F4	F5	F6
20	0.305	0.322	0.241	0.226	0.079	0.189
30	0.245	0.264	0.219	0.205	0.085	0.164
40	0.236	0.257	0.225	0.211	0.102	0.151
50	0.234	0.255	0.221	0.202	0.122	0.138

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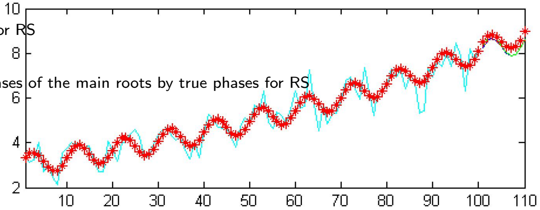

Average forecast errors, $h_j = \sin(2\pi j/17) + 0.5\sin(2\pi j/10)$, $r = 4$, $\sigma = 0.5$

L	F1	F2	F3	F4	F5	F6
20	1.061	1.057	0.281	0.249	0.091	0.221
30	0.678	0.676	0.239	0.199	0.095	0.169
40	0.622	0.622	0.250	0.208	0.117	0.161
50	0.613	0.610	0.251	0.199	0.132	0.147

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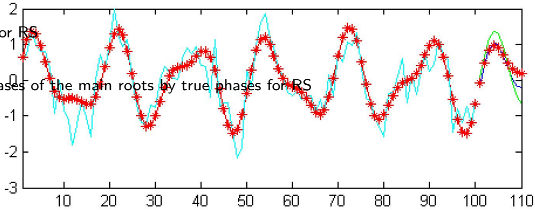

Average forecast errors, $h_j = \sin(2\pi j/6) + 0.5\sin(2\pi j/10)$, $r = 4$

σ	L	F1	F2	F3	F4	F5	F6
0.5	20	0.320	0.337	0.259	0.232	0.079	0.205
0.5	30	0.297	0.314	0.238	0.210	0.085	0.177
0.5	40	0.261	0.283	0.242	0.207	0.101	0.165
0.5	50	0.248	0.268	0.239	0.196	0.129	0.144
0.1	20	0.058	0.062	0.047	0.043	0.012	0.039
0.1	30	0.053	0.057	0.044	0.039	0.015	0.034
0.1	40	0.047	0.051	0.045	0.039	0.018	0.032
0.1	50	0.047	0.051	0.046	0.038	0.025	0.029

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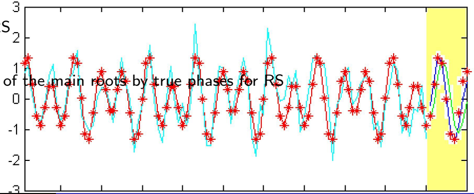

Conclusions

- Forecast error is decreasing as L is increasing
- Errors of ESPRIT LRF and SSA LRF are almost equal for $r = 2$
- SSA LRF is the best LRF (among considered LRF) for $r > 2$
- correction of the phases of the main roots by true phases in SSA LRF decrease the forecast error
- the error related to the coefficients of SSA LRF is smaller than the error from initial conditions for LRF. These errors are almost equal for $L = N/2$