Analytic moment and Laplace transform formulae for the quasi-stationary distribution of the Shiryaev diffusion on an interval

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This work is an investigation into quasi-stationarity of the classical Shiryaev diffusion restricted to an interval. Specifically, we study the solution $(R_t^r)_{t\geq 0}$ of the stochastic differential equation

$$dR_t^r = dt + R_t^r dB_t \text{ with } R_0^r = r \geq 0 \text{ fixed,} \tag{1}$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion (i.e., $\mathbb{E}[dB_t] = 0$, $\mathbb{E}[(dB_t)^2] = dt$, and $B_0 = 0$). The time-homogeneous Markov process $(R_t^r)_{t\geq 0}$ is a particular version of the generalized Shiryaev process. The latter has been first arrived at and studied by Prof. A.N. Shiryaev in his fundamental work [2, 3] on quickest change-point detection. This work is, too, inspired by applications of $(R_t^r)_{t\geq 0}$ in quickest change-point detection; see [1, 9, 8, 7]. The interest in $(R_t^r)_{t\geq 0}$ given by (1) is due to the fact that it is the *only* version of the generalized Shiryaev process with probabilistically nontrivial behavior in the limit as $t \to +\infty$, exhibited in spite of the distinct martingale property $\mathbb{E}[R_t^r - r - t] = 0$ for all $t \geq 0$ and $r \geq 0$. Moreover, the process is convergent regardless of whether the state space is (I) the entire half-line $[0, +\infty)$ with no absorption on the interior; or (II) the interval $[0, A]$ with absorption at a given level $A > 0$; or (III) the shortened half-line $[A, +\infty)$ also with absorption at $A > 0$ given. The limiting distribution in case (I) is called the stationary distribution, while that in cases (II) and (III) is referred to as *quasi*-stationary distribution. Cases (I), (II), and (III) have all been considered in the literature. See, e.g., [5, 6, 4, 11, 10]. However, this work's focus is on case (II) due to its significance in quickest change-point detection.

The specific contribution of this work pertaining to case (II) is two-fold: *(a)* obtain exact closed-form moment formulae for the quasi-stationary distribution; and subsequently use the moment formulae to *(b)* derive an exact formula for the Laplace transform of the quasi-stationary distribution. The moment formulae are obtained as an extension of the effort made earlier in [7] where the moment sequence was shown to satisfy a certain recurrence. This work solves the recurrence explicitly. We also compute the Laplace transform based on the obtained moment formulae.

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