Locally D-optimal Designs for Non-linear Models on the k-dimensional Ball

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1 Introduction

We want to construct (locally) D-optimal designs for a wide class of linear and non-linear multiple regression models where the design region is a k-dimensional unit ball $\mathbb{B}_k$.
This class of regression models was already investigated, for example, in Konstantinou, Biedermann and Kimber [1]. But they only focused on a one-dimensional bounded interval. Schmidt and Schwabe [4] considered the same class of models, but with k covariates on a k-dimensional cuboid. They found a way to divide this problem into k marginal sub-problems with only one covariate in the form like Konstantinou et al. [1].
This special class of regression problems contains, for example, Poisson regression, negative binomial regression and regression models with censored data (see Schmidt and Schwabe [4]).
In linear multiple regression models on the k-dimensional unit ball $\mathbb{B}_k$ it is known (see Pukelsheim [2]) that the D-optimal design consists of the vertices of an arbitrarily rotated k-dimensional regular simplex, whose vertices lie on the surface of the design region – the unit sphere $\mathbb{S}_{k-1}$. Here "regular" means that all edges of the simplex have the same length.
In our case using invariance and equivariance (see Radloff and Schwabe [3]) the multidimensional problem can be reduced only to a one-dimensional marginal problem, which is analogous to Konstantinou et al. [1].

2 Problem and Result

The focus is on multiple regression models where the design region is a k-dimensional unit ball $\mathbb{B}_k$. This means for every design point $\boldsymbol{x} \in \mathbb{B}_k$ the regression function $\boldsymbol{f} : \mathbb{B}_k \to \mathbb{R}^{k+1}$ is considered to be $\boldsymbol{x} \mapsto (1, x_1, \ldots, x_k)^\top$. The parameter vector $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)^\top$ is unknown and lies in the parameter space $\mathbb{R}^{k+1}$. This results in the linear predictor

$$\boldsymbol{f}(\boldsymbol{x})^\top \boldsymbol{\beta} = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k \ .$$

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For the underlying model the the one-support-point (or elemental) information matrix $\boldsymbol{M}(\boldsymbol{x},\boldsymbol{\beta})$ is assumed to be in the form

$$\boldsymbol{M}(\boldsymbol{x},\boldsymbol{\beta}) = \lambda\big(\boldsymbol{f}(\boldsymbol{x})^\top\boldsymbol{\beta}\big)\,\boldsymbol{f}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})^\top$$

with an intensity (or efficiency) function λ which only depends on the value of the linear predictor.
As in Konstantinou et al. [1] and Schmidt and Schwabe [4] the intensity function λ is assumed to satisfy the following four conditions:

(A1) λ is positive on $\mathbb{R}$ and twice continuously differentiable.

(A2) λ' is positive on $\mathbb{R}$.

(A3) The second derivative u'' of $u = \frac{1}{\lambda}$ is injective on $\mathbb{R}$.

(A4) The function $\frac{\lambda'}{\lambda}$ is a non-increasing function.

Then the (locally) D-optimal design consists of a pole of the unit ball and the vertices of a $(k-1)$-dimensional regular simplex inscribed in the cut set of a plain, which is orthogonal to the polar axis, and the unit sphere.
For $(\beta_1,\ldots,\beta_k)^\top \neq (0,\ldots,0)^\top$ the pole is the unique point on the sphere maximising the intensity, and the distance of the cutting plain to the pole depends on the magnitude of $||(\beta_1,\ldots,\beta_k)^\top||$.

References

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