Abstracts of the Ninth Workshop on Simulation (2018) 1-2

Optimal Designs for Count Data with Random Parameters

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1 Introduction

Count data arises in experiments, where the number of objects or occurrences of events of interest is observed. Frequently, the Poisson model is used to model such data, in which the expected value of the Poisson distributed response variable is linked to a linear predictor consisting of covariates and unknown model parameters. Assuming a Gamma distributed random effect for each statistical unit, we obtain the Poisson-Gamma model as a generalization of the Poisson model. We note that there may be repeated measurements for each statistical unit.

The estimates of the unknown model parameters depend on the choice of the covariates. In order to obtain the most accurate parameter estimates, we determine optimal designs, which specify the optimal values and frequencies of the covariates (see [4]). With such designs the number of experimental units can be reduced, leading to a reduction of experimental costs.

For the Poisson model Rodríguez-Torreblanca and Rodríguez-Díaz [2] determined D- and c-optimal designs for the case of one covariate and Russell et al. [3] derived D-optimal designs for the case of multiple covariates. In the context of intelligence testing Graßhoff et al. [1] considered the Poisson-Gamma model with one measurement per statistical unit and computed D-optimal designs for a binary design region.

2 Formulation of the problem

We consider n statistical units, for example groups or individuals, for each of which m experiments with response variables Y_{ij}, $i = 1, \dots, n$, $j = 1, \dots, m$, are performed. To each statistical unit a Gamma distributed block effect $\Theta_i \sim \gamma(a, b)$ is assigned. The probability density function of the Gamma distribution $\gamma(a, b)$, $a, b > 0$, is given by $f_\gamma(\theta) = \frac{b^a}{\Gamma(a)} \cdot \theta^{a-1} \cdot e^{-b\theta}$ for $\theta > 0$, where $\Gamma(a)$ denotes the Gamma function.

We assume that given $\Theta_i = \theta_i$ the random variables Y_{ij} are independent Poisson distributed with parameter λ_{ij}. The expected value λ_{ij} is related via the canonical link function to the linear predictor, which consists of a fixed effects term $\boldsymbol{f}(\boldsymbol{x}_{ij})^T\boldsymbol{\beta}$ and an additive random effect v_i:

$$\ln(\lambda_{ij}) = \boldsymbol{f}(\boldsymbol{x}_{ij})^T\boldsymbol{\beta} + v_i$$

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The values of the covariates $\boldsymbol{x}_{ij}$ can be chosen from a design region $\mathscr{X} \subset \mathbb{R}^k$. The vector $\boldsymbol{f} = (1, f_1, \ldots, f_{p-1})$ consists of known regression functions and the vector $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_{p-1})^T$ is the unknown parameter vector. The random effect is given by $v_i = \ln(\theta_i)$. It follows:

$$\lambda_{ij} = \theta_i \cdot e^{\boldsymbol{f}(\boldsymbol{x}_{ij})^T \boldsymbol{\beta}}$$

Since the Poisson and Gamma distribution are conjugate distributions, the probability density function of $\boldsymbol{Y_i} = (Y_{i1}, \ldots, Y_{im})$ and thus the information matrix can be derived analytically. Optimal designs are based on the optimization of a function of the information matrix, for example, for D-optimality the determinant of the information matrix is maximized with respect to the design. Since the model is nonlinear, the optimal designs depend on the unknown parameters and are therefore called locally optimal.
We investigate the relations between the information matrices for the Poisson and Poisson-Gamma model and show that the D-optimality criterion for the Poisson-Gamma model is equivalent to a combined weighted optimality criterion of D-optimality and D_s-optimality for $\beta_1, \ldots, \beta_{p-1}$ for the Poisson model. Moreover, we determine the D-optimal designs for the Poisson-Gamma model, obtaining the D_s-optimal designs for the Poisson model as a special case.
For linear optimality criteria like L- and c-optimality we show that the optimal designs in the Poisson and Poisson-Gamma model coincide, which facilitates the search for optimal designs.

References

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