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Universal modification of vector weighted method of correlated sampling with finite computation cost

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Consider the system of the 2nd kind integral equations with parameter λ:

$$\phi_i(x,\lambda) = \sum_{j=1}^{m} \int\limits_X k_{ij}(x,y,\lambda)\phi_j(y,\lambda)dy + h_i(x,\lambda), \quad i = 1, ..., m, \tag{1}$$

or $\Phi_\lambda = \mathbf{K}_\lambda \Phi_\lambda + H_\lambda$, in the vector form, where $\lambda \in \Lambda = \{\lambda_1, ..., \lambda_s\}$, $H_\lambda^{\mathrm{T}} = (h_1(\lambda), \ldots, h_m(\lambda)) \in L_\infty$, $\mathbf{K}_\lambda \in [L_\infty \to L_\infty]$, $||H_\lambda||_{L_\infty} = \text{vrai} \sup\limits_{i,x} |h_i(x,\lambda)|$. We assume that the spectral radius is $\rho(\mathbf{K}_{\lambda,1}) < 1$ where $\mathbf{K}_{\lambda,1}$ is the operator obtained from the operator $\mathbf{K}_{\lambda,1}$ by replacement $k_{ij}(x,y,\lambda) \to |k_{ij}(x,y,\lambda)|$ and $\rho(\mathbf{K}_{\lambda,1}) = \underline{\lim} ||\mathbf{K}_{\lambda,1}^n||_{L_\infty}^{1/n} = \inf ||\mathbf{K}_{\lambda,1}^n||_{L_\infty}^{1/n}$. The Neumann series for (1) converges if the inequality $||\mathbf{K}_{\lambda,1}^{n_0}|| < 1$ holds for some $n_0 \geq 1$ and $\forall \lambda \in \Lambda$ where $||\mathbf{K}_{\lambda,1}|| = \sup\limits_{x,i} \sum\limits_{j=1}^{m} \int |k_{ij}(x,y,\lambda)|dy$.

Let us consider a Markov chain of collisions $\{x_n\}, (n = 0, \ldots, N)$ with the transition density $p(x,y)$ and $\rho(B_p) < 1$, where B_p is an integral operator with the kernel $p(x,y)$ (in particular, for $p(x) \geq \varepsilon > 0$). A standard vector Monte Carlo collision estimator is constructed for the value $\Phi_\lambda(x) = \mathrm{E}\boldsymbol{\xi}_x(\lambda)$ on the base of relations

$$\boldsymbol{\xi}_x(\lambda) = H_\lambda(x) + \delta_y Q(x,y,\lambda)\boldsymbol{\xi}_y(\lambda), \tag{2}$$

where $Q(x,y,\lambda) = K(x,y,\lambda)/p(x,y)$ is the matrix weight and δ_x is the indicator of chain non-breaking in the transition $x \to y$. The variance $\mathrm{V}\boldsymbol{\xi}_x(\lambda)$ depends on the matrix of second moments $\Psi(x,\lambda) = \mathrm{E}(\boldsymbol{\xi}_x(\lambda)\boldsymbol{\xi}_x^{\mathrm{T}}(\lambda))$ which satisfies the following equation ([2])

$$\Psi(x,\lambda) = \chi(x,\lambda) + \int\limits_X \frac{K(x,y,\lambda)\Psi(y,\lambda)K^{\mathrm{T}}(x,y,\lambda)}{p(x,y)}dy, \tag{3}$$

or $\Psi_\lambda = \chi_\lambda + \mathbf{K}_{\lambda,p}\Psi_\lambda$, where $\chi_\lambda = H_\lambda\Phi_\lambda^{\mathrm{T}} + \Phi_\lambda H_\lambda^{\mathrm{T}} - H_\lambda H_\lambda^{\mathrm{T}}$. This equation is considered in the space $\mathbf{L}_\infty$. Denote the operator obtained from $\mathbf{K}_{\lambda,p}$ by replacing the kernels by their absolute values by $\mathbf{K}_{\lambda,p,1}$. It is known ([2]) that if $\rho(\mathbf{K}_{\lambda,p,1}) < 1$ then $\Psi(x,\lambda) = \mathrm{E}(\boldsymbol{\xi}_x(\lambda)\boldsymbol{\xi}_x^{\mathrm{T}}(\lambda))$ is a solution of the equation (3) and $\Psi_\lambda \in \mathbf{L}_\infty$. The corresponding Monte Carlo algorithms (see (2)) were naturally named the method

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of similar trajectories (MST) [1] or the weighted method of correlated sampling [2]. The use of such methods allows one to construct estimators of functionals on a single Markov chain simultaneously for a given range of parameters (Λ) and to study effectively the results dependence on the parameters of the system, in particular, for their small variations [1].

If the spectral radius is $\rho(\mathbf{K}_{\lambda,p,1}) > 1$, then the elements of $\mathrm{V}\boldsymbol{\xi}(\lambda)$ can be infinitely large and the use of the weight estimator $\boldsymbol{\xi}_x(\lambda)$ for the calculation of the $\Phi(x,\lambda)$ is not expedient. Note that the estimation of the value $\rho(\mathbf{K}_{\lambda,p,1})$ for real problems demands a separate and labor-consuming theoretical study. The standard way to decrease the value $\mathrm{V}\boldsymbol{\xi}_x(\lambda)$ is the use of the following modification of the estimator $\boldsymbol{\xi}_x(\lambda)$ with additional trajectory branching (randomized splitting)

$$\boldsymbol{\zeta}_{x_0}(\lambda) = H_\lambda(x_0) + \delta_{x_1}\frac{Q^{(1)}(x_0,x_1,\lambda)}{\mathrm{E}\nu_1}\sum_{i=1}^{\nu_1}\boldsymbol{\zeta}_{x_1}^{(i)}(\lambda), \tag{4}$$

$$\boldsymbol{\zeta}_{x_{n-1}}(\lambda) = H_\lambda(x_{n-1}) + \delta_{x_n}\frac{Q^{(1)}(x_{n-1},x_n,\lambda)}{\mathrm{E}\nu_n}\sum_{i=1}^{\nu_n}\boldsymbol{\zeta}_{x_n}^{(i)}(\lambda), \tag{5}$$

where $\boldsymbol{\zeta}_{x_n}^{(\cdot)}(\lambda)$ are independent realizations of $\boldsymbol{\zeta}_{x_n}(\lambda)$ and $Q^{(1)}(x_{n-1},x_n)$ corresponds to $\boldsymbol{K}$ with $|k_{ij}(x,y,\lambda)|$.

If we introduce $\{A\}_{ij} = a_{ij}$ for an arbitrary element of the matrix A and

$$\mathrm{E}\nu_1 = \max_\lambda \sup_i \sum_{j=1}^m \{Q^{(1)}(x_0,x_1,\lambda)\}_{ij},\ \mathrm{E}\nu_n = \max_\lambda \sup_i \sum_{j=1}^m \left\{\frac{Q_{n-1}^{(1)}Q^{(1)}(x_{n-1},x_n,\lambda)}{\mathrm{E}\nu_1 ... \mathrm{E}\nu_{n-1}}\right\}_{ij}, \tag{6}$$

then, as in [3] it is easy to check that $\mathrm{E}\boldsymbol{\zeta}_x(\lambda) = \Phi_\lambda(x)$ and $\mathrm{E}(\boldsymbol{\zeta}_x(\lambda)\boldsymbol{\zeta}_x^{\mathrm{T}}(\lambda)) \in \mathbf{L}_\infty$ if only $\rho(\mathbf{K}_{\lambda,1}) < 1$ and $\rho(B_p) < 1$. The last inequalities also ensure that the mean computer simulation time for calculation of one sample value of $\boldsymbol{\zeta}_x$ is also bounded.

Numerical study of the efficiency of the presented weighted method of correlated sampling with branching in comparison with "analog" modelling for every parameter $\lambda \in \Lambda$ was carried out on the example of the standard problem of transfer theory with allowance for polarization.

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References

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